NOTE ON THE CONVERGENCE OF WEIGHTED TRIGONOMETRIC SERIES*

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1. *Introduction.* Let $f(x)$ be a function continuous for all values of x, and of period 2π. Let $T_n(x)$ be a trigonometric sum of the nth order.† If $T_n(x)$ is determined, among all such sums, by the condition that the value of the integral

$$\int_0^{2\pi} [f(x) - T_n(x)]^2 dx$$

shall be a minimum, it becomes the partial sum of the Fourier series for $f(x)$. The problem can be generalized by taking, as the quantity to be reduced to a minimum, the integral

$$\int_0^{2\pi} \rho(x)[f(x) - T_n(x)]^2 dx, \tag{1}$$

where $\rho(x)$, indicating the *weight* to be attached to different values of the argument, is a function of x, likewise of period 2π, and positive for all values of x. There is a considerable body of literature bearing more or less directly on the generalized problem. This literature owes its inspiration largely to the researches of Tchebychef;‡ particular mention should also be made of a classical memoir by Gram.§

The purpose of the following paragraphs is to discuss the convergence of $T_n(x)$ toward the value $f(x)$, as n becomes infinite. The method is one which I have used recently in connection with the corresponding problem in which the weight is constantly equal to unity, and the square of the error is replaced by a power with a different exponent. The

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† The words "of the nth order" will be understood throughout to mean "of the nth order at most."

‡ Cf., e.g., H. Burkhardt, *Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik*, JAHRESBERICHT DER VEREINIGUNG, vol. 10, Heft 2 (1908), pp. 823 ff.

§ J. P. Gram, *Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate*, JOURNAL FÜR MATHEMATIK, vol. 94 (1883), pp. 41–73.