A GENERALIZATION OF NORMAL CONGRUENCES OF CIRCLES *

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1. *Introduction.* A congruence of circles in three-dimensional space is said to be *normal* if every circle of the congruence is normal to three surfaces. Normal congruences have long been studied,† and one of their principal properties is expressed in the theorem that if a variable circle C is normal to the three fixed surfaces S_1, S_2, S_3 at the points P_1, P_2, P_3 respectively, and if the point P_4 is determined by a real constant cross ratio with P_1, P_2, P_3, then as C varies the point P_4 traces a surface which is also orthogonal to C.

It is the purpose of the present note to consider a type of congruence to which we shall give the name of isogonal congruence and which is a generalization of the notion of normal congruence. A congruence of circles is said to be *isogonal* if every circle of the congruence cuts three surfaces at equal angles in such a way that when the circle is inverted into a straight line L, the tangent planes to the corresponding surfaces at their points of intersection with L are all parallel. That is, every sphere through a circle of the congruence cuts at equal angles the three surfaces at their points of intersection with that circle. It is to be noted that the term isogonal might well be given to a still larger type of congruence of circles, but in the present paper the term will be used only in the restricted sense indicated.

We shall prove (Theorem III) that if a congruence is isogonal there are not merely three surfaces but a one-parameter family of surfaces which have the isogonal property, and all the surfaces of the family can be obtained as in the case of normal congruences.

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† By Ribaucour, Darboux, Bianchi, Eisenhart, and Coolidge, among others. Detailed references are given by Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, Chap. XV.