A REVISION OF THE BERNOULLIAN AND EULERIAN FUNCTIONS *

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1. *Introduction.* To secure simplicity and uniformity in the derivation of relations between the Bernoullian and Eulerian *functions* (not to be confused with the *numbers* of the same name) occurring most frequently in applications we take as fundamental a set of four functions instead of the usual pair, and from these by easy substitutions obtain the values of the functions defined by other writers.†

Following Lucas ‡ we use the even suffix notation for the numbers B, G, E, R of Bernoulli, Genocchi, Euler and Lucas; with the exceptions $B_1 = -\frac{1}{2}$, $G_1 = 1$, the numbers of odd rank vanish, and $B_0 = 1$, $G_0 = 0$, $E_0 = 1$, $R_0 = \frac{1}{2}$. Unless otherwise stated n is an arbitrary integer $\geqq 0$. The "representative" or umbral calculus of Blissard § is used throughout, so that the nth power a^n of the umbra a represents the ordinary a_n. The letters x, y, z, u, v denote ordinary algebraic quantities, or ordinaries; $a, b, c, B, E, G, R, \varphi, \psi, \lambda, \mu$ are umbræ. The umbral sin ax, cos ax are defined by the series, assumed absolutely convergent for some $|x| > 0$,

$$\sin ax = \sum_0^\infty (-1)^n a^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_0^\infty (-1)^n a_{2n+1} \frac{x^{2n+1}}{(2n+1)!},$$

$$\cos ax = \sum_0^\infty (-1)^n a^{2n} \frac{x^{2n}}{(2n)!} = \sum_0^\infty (-1)^n a_{2n} \frac{x^{2n}}{(2n)!};$$

the umbral multinomial theorem gives the expansion of $(ax + by + \cdots + cz)^n$ in either of the identical forms

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† Accounts of the history and notations of these functions are contained in papers by Worpitzky, CRELLE'S JOURNAL, vol. 94 (1883), pp. 203–232, and Glaisher, QUARTERLY JOURNAL, vol. 29 (1898), pp. 1–169; vol. 42 (1911), pp. 86–157.

‡ *Théorie des Nombres,* Chap. XIV.

§ QUARTERLY JOURNAL, vols. 6–9 (1863–1867); cf. also Lucas, loc. cit., Chap. XIII.