IMPOSSIBILITY OF RESTORING UNIQUE FACTORIZATION IN A HYPER-COMPLEX ARITHMETIC

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1. *Introduction.* Most numbers $a + be$, where a and b are integers and $e^2 = 0$, admit of several factorizations into indecomposable numbers. It is proved in § 3 that we cannot restore unique factorization by defining hypercomplex ideals analogous to algebraic ideals, nor (§ 4) by the introduction of any sort of ideals obeying the laws of arithmetic. L. G. du Pasquier* has made statements, omitting proofs, concerning the failure of unique factorization after introducing ideals, apparently meaning those analogous to algebraic ideals.

2. *Hypercomplex Integers.* Consider the hypercomplex numbers $x = a + be$ with rational coordinates a, b, where $e^2 = 0$. Thus $(x - a)^2 = 0$. This quadratic equation has integral coefficients if and only if a is integral. As our integral hypercomplex numbers we shall take those of an infinite system of numbers $a + be$, where a is integral and b rational, such that the system has a basis† 1, ce, i.e., is composed of their linear combinations with integral coefficients. Since we may take ce as a new unit e whose square is zero, we may assume that $a + be$ is integral if and only if a and b are both integers.

* Vierteljahrsschrift, Zürich, vol. 54 (1909), pp. 116–148. L'Enseignement, vol. 17 (1915), pp. 340–3; vol. 18 (1916), pp. 201–260. Nouvelles Annales, (4), vol. 18 (1918), pp. 448–461. Comptes Rendus du Congrès International (Strasbourg), 1921.

† We obtain uninteresting results if we omit the assumption of a basis and call $a + be$ integral if a is integral and b rational. It is a unit if $a = \pm 1$. If $r \neq 0$, $\rho = r + se$ is "associated" with its product r by the unit $1 - es/r$. Hence the classes of associated numbers whose real coordinates are not zero are in (1, 1) correspondence with the real integers and obey the laws of divisibility of integers. But se is associated only with $\pm se$. Now te is divisible by every $r + se$, $r \neq 0$, the quotient being et/r.