TWO THEOREMS ON MULTIPLE INTEGRALS*

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1. *Introduction.* The multiple integrals in question here were discussed by Poincaré (ACTA MATHEMATICA, vol. 9, p. 321) in a paper in which he defined them and derived their integrability conditions. In this note we give two theorems, which simplify the derivation of these conditions, and enable us to express the integral of an integrable function over an open region as an integral of one order lower over the boundary of this region. The second theorem was suggested by a special case proved by Professor Eisenhart.

2. *First Theorem.* Our first theorem is

$$\int n \int \sum A_{i_1\ldots i_n} dx_{i_1}\cdots dx_{i_n} \qquad (1)$$
$$\frac{1}{m-n}\int n+1 \int \sum \frac{\partial A_{i_1\ldots i_n}}{\partial x_{i_{n+1}}} dx_{i_{n+1}} dx_{i_1}\cdots dx_{i_n},$$

where the n (dimensional) region of integration for the left member is the boundary of the $n+1$ region for the right member, and the summations apply to all n or $n+1$ permutations of the m subscripts of our fundamental space. The theorem is proved by adding equations of the form

$$\int n \int A_{1\ldots n} dx_1\cdots dx_n = \int n+1 \int \frac{\partial A_{1\ldots n}}{\partial x_{n+1}} dx_{n+1} dx_1 \cdots dx_n, \qquad (2)$$

which may be established when the A's are functions of m variables by methods similar to those ordinarily used to prove (2) in n-space. The equation (1) shows that the set of equations

$$\sum_j (-1)^{j-1} \frac{\partial A_{i_1\ldots i_{j-1} i_{j+1}\ldots i_{n+1}}}{\partial x_{i_j}} = 0 \qquad (3)$$

is a necessary and sufficient condition for the vanishing of the left member of (1) for all closed regions of integration, i.e.,

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