CONDITION THAT A TENSOR BE THE CURL OF A VECTOR *

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It is the purpose of this note to establish the following theorem.

THEOREM. *A necessary and sufficient condition that a covariant skew-symmetric tensor A_{ij} in a space of any order n be expressible in terms of n functions φ_i in the form*

$$A_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} \tag{1}$$

is that

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0, \quad (i, j, k = 1, \cdots, n). \tag{2}$$

Consider first the case of 3-space. If φ_2 and φ_3 are any two functions such that

$$A_{23} = \frac{\partial \varphi_2}{\partial x^3} - \frac{\partial \varphi_3}{\partial x^2},$$

the conditions of integrability of

$$\frac{\partial \varphi_1}{\partial x^2} = \frac{\partial \varphi_2}{\partial x^1} + A_{12}, \qquad \frac{\partial \varphi_1}{\partial x^3} = \frac{\partial \varphi_3}{\partial x^1} + A_{13}$$

are satisfied in consequence of (2), and the theorem is established for 3-space.

Now we show that, if the theorem is true for n-space, it is true for $(n+1)$-space. On this assumption equations (1) hold for $i, j = 1, \cdots, n$. For a particular i and j and for $k = n+1$, equation (2) may be written in the form

$$\frac{\partial}{\partial x^i}\left(A_{jn+1} - \frac{\partial \varphi_j}{\partial x^{n+1}}\right) = \frac{\partial}{\partial x^j}\left(A_{in+1} - \frac{\partial \varphi_i}{\partial x^{n+1}}\right).$$

Hence a function φ_{n+1} is defined by the equations

$$A_{in+1} = \frac{\partial \varphi_i}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^i}, \qquad A_{jn+1} = \frac{\partial \varphi_j}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^j}. \tag{3}$$

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