NOTE ON EULER'S φ–FUNCTION

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Two correspondents have recently called my attention to the fact that the supposed proof of the following theorem, which I gave some years ago,* is not adequate:

THEOREM I. *For a given number n, the equation* $\varphi(x) = n$ *either has no solution or it has at least two solutions.*

So far I have been unable to supply a proof of the theorem, though it seems probable that it is correct. I am therefore compelled to allow it to stand in the status of a conjectured or empirical theorem.

Let us examine the hypothesis that there exists a value ν of n such that $\varphi(x) = \nu$ has one and just one solution. It is easy to derive certain necessary properties of x. In the first place, x is even, since otherwise $2x$ would also be such that $\varphi(2x) = \nu$. Again, x is divisible by 4, since otherwise $\varphi(x/2)$ would be equal to ν. Let us then denote the value of x by $4s$. We shall prove the following theorem.

THEOREM II. *If* $4s$ *has the factor* $p_0^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, *where* p_0 $(= 2)$, $p_1, p_2, \cdots, p_k$ *are distinct prime numbers, and if the quotient of* $4s$ *by this factor is prime to the factor, and if* $p_0^{\gamma_0}p_1^{\gamma_1} \cdots p_k^{\gamma_k} + 1$ *is a prime number* q, *where for a given* i, $0 < \gamma_i < \alpha_i$, *then* $4s$ *has the factor* q^2.

The proof is almost immediate. For we have

$$\varphi(2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \varphi(2^{\alpha_0-\gamma_0}p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2} \cdots p_k^{\alpha_k-\gamma_k}q),$$

so that we should have two solutions of the equation $\varphi(4s) = \nu$ unless s contains the factor q. Similarly, it may be shown that s contains the factor q^2, since otherwise the first power of q could be omitted by appropriately raising certain (or all)

* This BULLETIN, vol. 13 (1907), p. 241. The theorem is also stated as an exercise in my *Theory of Numbers*, p. 36; it was its presence here that led each correspondent to the discovery of the error.