

## NOTE ON AN IRREGULAR EXPANSION PROBLEM\*

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One of the simplest expansion problems depending on a set of boundary conditions of the type which Birkhoff has called *irregular*, is that associated with the system

$$(1) \quad \frac{d^3u}{dx^3} + \rho^3u = 0,$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(\pi) = 0.$$

Let  $u_1(x)$ ,  $u_2(x)$ ,  $\dots$  be the characteristic functions of this differential system. The writer showed, a number of years ago,† that the formal expansion of an arbitrary continuous function  $f(x)$  in a series of the form

$$(2) \quad a_1u_1(x) + a_2u_2(x) + \dots$$

will ordinarily be divergent, even if  $f(x)$  satisfies conditions which would insure the convergence of its development in terms of the characteristic functions of a regular system; and it may diverge even if  $f(x)$  is analytic throughout the interval  $(0, \pi)$ . A subsequent paper by Hopkins‡ specified a restricted class of analytic functions to which  $f(x)$  must belong, if the expansion is to converge uniformly to the desired value. He showed that the necessary condition thus obtained, if supplemented by certain secondary hypotheses, is also sufficient. He did not discuss the question whether the formal expansion, if uniformly convergent at all, must necessarily have  $f(x)$  for its sum. His analysis, however, makes it possible to recognize without difficulty that this is the case. It is

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† D. Jackson, *Expansion problems with irregular boundary conditions*, PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 51 (1915-16), pp. 383-417; see pp. 384-393.

‡ J. W. Hopkins, *Some convergent developments associated with irregular boundary conditions*, TRANSACTIONS OF THIS SOCIETY, vol. 20 (1919), pp. 245-259.