This is the surface of revolution of a parabola of latus rectum $8m$ about its directrix. A similar result was obtained by Flamm* who considered the surface, in euclidean three-space, for which the linear element is given by (2) for $u_2 = \pi/2$.

PRINCETON UNIVERSITY,
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A COVARIANT OF THREE CIRCLES.

BY PROFESSOR A. B. COBLE.

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Dr. J. L. Walsh † has stated the following theorem.

THEOREM. *If the double ratio,* $(z_1, z_3 \mid z_2, z)$, *of the four points* z_1, z_2, z_3, z *in the complex plane is a real number* λ, *then as the points* z_1, z_2, z_3 *run over the circles* C_1, C_2, C_3 (*and their interiors*) *respectively, the locus of* z *is a circle* (*and its interior*).

This locus is evidently a covariant, under the inversive group, of the three given circles, which is rational in λ. We find in (8) its equation and incidentally prove the theorem.

In conjugate coordinates $z, \bar{z}$, a circle is

$$C_1(z) = a_1 z\bar{z} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + b_1 = 0,$$

where a_1, b_1 are real, and $\alpha_1, \bar{\alpha}_1$ are conjugate imaginary. The bilinear invariant of two circles $C_1(z), C_2(z)$ is

$$[C_1, C_2] = \alpha_1\bar{\alpha}_2 + \alpha_2\bar{\alpha}_1 - a_1 b_2 - a_2 b_1.$$

It vanishes when the two circles are orthogonal. When they coincide it becomes $[C_1 C_1] = 2(\alpha_1\bar{\alpha}_1 - a_1 b_1)$. This vanishes when C_1 is a *point circle*, i.e. one whose equation is

$$(1) \qquad P_{z_i}(z) = (z - z_i)(\bar{z} - \bar{z}_i) = 0.$$

It is easily verified that

$$[C_1, P_{z_i}(z)] = -C_1(z_i); \quad [P_{z_i}(z), P_{z_k}(z)] = -P_{z_i}(z_k) = -P_{z_k}(z_i).$$

The two point circles of the pencil $C(z) + \mu K(z) = 0$ are determined by

$$[C + \mu K, C + \mu K] = [C, C] + 2\mu\,[CK] + \mu^2\,[KK] = 0.$$

* PHYSIK. ZEITSCHR., vol. 17 (1916), p. 449.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 22 (1921), p. 101. The geometric proof of this theorem given by Dr. Walsh is very complicated. The method of proof followed here is considered by Dr. Walsh (loc. cit.,