# ON THE FOURIER COEFFICIENTS OF A CONTINUOUS FUNCTION.

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It is well known that when

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier expansion of a function $f(\theta)$ which is real and continuous for $0 \leqq \theta \leqq 2\pi$, then $\Sigma(a_n{}^2 + b_n{}^2)$ converges. Here the exponent 2 cannot in general be replaced by a smaller one; in fact, Carleman* has constructed an example of a continuous $f(\theta)$ where $\Sigma(a_n{}^{2-2\delta} + b_n{}^{2-2\delta})$ diverges for any $\delta > 0$, and this example has been simplified by Landau.†

In the present note it will be shown that, *given any single-valued real function $\varphi(x)$, subject only to the condition that $\varphi(x)$ becomes infinite as $x$ becomes infinite, there exists a real continuous function $f(\theta)$ whose Fourier coefficients $a_n$, $b_n$ make the series*

$$\sum (a_n{}^2 + b_n{}^2)\varphi\left(\frac{1}{a_n{}^2 + b_n{}^2}\right)$$

*divergent.* Assuming $\varphi(x) = x^\delta$, where $\delta > 0$, and observing that $(a^2 + b^2)^{1-\delta} < a^{2-2\delta} + b^{2-2\delta}$, we have the particular result referred to above.

If we denote by $f_1(\theta)$ the function conjugate to $f(\theta)$, and write $z = e^{\theta i}$, $F(z) = f(\theta) + if_1(\theta)$, the Fourier expansion of $F(z)$ is $\Sigma_0^\infty c_n z^n$, where $c_0 = a_0/2$, $c_n = a_n - ib_n$ $(n > 0)$. Our statement will be proved by constructing a function $F(z)$ continuous for $|z| = 1$ and such that $\Sigma|c_n|^2\varphi(1/|c_n|^2)$ diverges. This will be done by means of the following result due to Hardy and Littlewood‡ and used by Landau, loc. cit., for a different purpose:

* T. Carleman, *Ueber die Fourierkoeffizienten einer stetigen Funktion*, ACTA MATH., vol. 41 (1918), pp. 377–384.

† E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147–153.

‡ G. H. Hardy and J. E. Littlewood, *Some problems of diophantine approximation*, ACTA MATH., vol. 37 (1914), pp. 155–239. See p. 220.