Then we may take n so great that $\xi_i^{(n)}$ contains every function in Σ_i, and then, since $\lambda_i^{(n)}$ is the least norm, we must have $\lambda_i^{(n)} < \epsilon$, and consequently $\lambda_i < \epsilon$. Hence

THEOREM IV. *A necessary and sufficient condition that a normalized system* $[\varphi]$ *be essentially linearly dependent is that* $\lambda_i = 0$ *for some* i.

Theorems II and IV give

THEOREM V. *A necessary and sufficient condition that a system* $[\varphi]$ *have an adjoint is that it be essentially linearly independent.*

THE UNIVERSITY OF OREGON,
November, 1919.

ON CERTAIN RELATED FUNCTIONAL EQUATIONS.

BY DR. W. HAROLD WILSON.

(Read before the American Mathematical Society December 27, 1917.)

§ 1. *Introduction.*

THIS paper treats of the relationships which exist between certain functional equations. In § 2, the equations

$$(1) \qquad S(x-y) = S(x)C(y) - C(x)S(y),$$

and

$$(2) \qquad C(x-y) = C(x)C(y) - k^2S(x)S(y)$$

are considered individually and as a system. It is shown that (1) and (2) have their solutions in common if $C(x)$ is an even function and $S(x) \not\equiv 0$. As a consequence, it is shown that if $k \neq 0$, then

$$S(x) = [F(x) - F(-x)]/2k, \text{ and } C(x) = [F(x) + F(-x)]/2,$$

where $F(x+y) = F(x)F(y)$. If $k = 0$ and $S(x) \not\equiv 0$, $C(x) \equiv 1$ and

$$S(x+y) = S(x) + S(y).$$

The work at this point is very closely allied to that of