ON THE NUMBER OF REPRESENTATIONS OF $2n$ AS A SUM OF $2r$ SQUARES.

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1. OWING possibly to its connection with X-ray analyses of crystal structure, interest in the problem of representing an integer as a sum of integral squares has recently revived. We shall first summarize briefly so much of what is known of the problem as will put the formulas established below in their proper light. Let $N(n, 2r)$ denote the total number of decompositions of n into a sum of $2r$ squares. Then, for $r = 1, 2, 3, 4$, the complete results concerning $N(n, 2r)$ are implicit in sections 40, 41, 42, 65 of the Fundamenta Nova. Jacobi, however, left the explicit statement of all but one of his results to others. When $r \not> 4$, $N(n, 2r)$ is expressible in terms of the divisors of n alone. By arithmetical methods, independently of elliptic functions, Eisenstein* proved some of Jacobi's results, showed how the rest might be obtained from his own theorems, and proved that, for $r > 4$ and n general, $N(n, 2r)$ can not be expressed in terms of the divisors of n alone. Letting $\xi_s'(n)$ denote the excess of the sum of the sth powers of all those divisors of n that are of the form $4k + 3$ over the like sum for all divisors of the form $4k + 1$, and $\zeta_s(n)$ the sum of the sth powers of all the divisors, Eisenstein stated a notable exception to his general theorem; showing that at least once, when n is suitably chosen, $N(n, 2r)$, for $r > 4$, may be expressed in terms of $\xi_s'(n)$, or $\zeta_s(n)$. E.g., $N(4k + 3, 10) = 12\xi_4'(4k + 3)$; or what may be shown is ultimately the same thing:† $N(8k + 6, 10) = 204\xi_4'(4k + 3)$. Liouville derived a similar result for $N(2m, 12)$ in terms of $\zeta_5(2m)$. He used for this purpose certain remarkable formulas‡ which, however, he did not prove, and which it is the

* *Crelle*, vol. 35 (1847), p. 135.

† Either result follows from the other on applying a transformation of the second order to the theta equivalent of the appropriate Liouville formula of the kind in § 2.

‡ *Jour. des Math.* (2), vol. 6 (1861); two papers, p. 233, 369 et seq. The formulas for proper representations may be proved similarly to those in this paper.