THE DERIVATIVE OF A FUNCTIONAL.

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IN his book on Integral Equations* Volterra has given a definition of the derivative of a functional and has stated somewhat restricted conditions under which the variation can be expressed as a linear integral. In the present paper it is shown that, under more general conditions, the variation is a linear functional in the sense of Riesz* and, therefore, a Stieltjes integral. This theorem is assumed as a condition in a paper by Fréchet.† Let

$$F[\overset{b}{\underset{a}{f(x)}}]$$

denote a functional F of a continuous function $f(x)$ $(a \leqq x \leqq b)$. With Volterra we shall consider only continuous functions. Let us denote the first variation by

$$D(f\,;\varphi) = \lim_{\epsilon \doteq 0} \frac{1}{\epsilon} \{F[f + \epsilon\varphi] - F[f]\}.$$

In place of Volterra's four conditions we take the two following:

I. $F[f]$ satisfies the Cauchy-Lipschitz condition, namely that we can find a number M such that

$$|F[f_1] - F[f_2]| \leqq M \max|f_1(x) - f_2(x)|.$$

II. The first variation $D(f'\,;\varphi)$ exists for all continuous φ, and all continuous f' in the neighborhood of f; that is to say that a number $\eta > 0$ can be found so that the variation exists so long as

$$\max|f'(x) - f(x)| \leqq \eta.$$

Under these conditions the variation is a linear functional, and therefore a Stieltjes integral,

$$D(f\,;\varphi) = \int_a^b \varphi(x)d\alpha(x).$$

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* V. Volterra, Equations Intégrales, p. 12 et seq. F. Riesz, *Annales de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 9.

† M. Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 135.