ON THE EVALUATION OF THE ELLIPTIC TRANSCENDENTS η_2 AND η_2'.

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WITH Halphen* write

$$f(s) = 4s^3 - g_2 s - g_3 = 4(s - e_1)(s - e_2)(s - e_3) = S,$$

where e_2 is a real quantity and e_1, e_3 are conjugate imaginaries such that $e_1 - e_3$ is a positive pure imaginary. The discriminant $\Delta = g_2^3 - 27g_3^2$ is here negative. The elliptic differential equation in this notation is $ds^2/du^2 = 4s^3 - g_2 s - g_3$, whose integral is

$$u = \int_s^\infty \frac{ds}{\sqrt{f(s)}}.$$

If the lower limit of this integral be considered as a function of the integral, that is of u, we may write $s = \wp u$.

As is well known, the quantity ω_2 is defined by

$$(1) \qquad \omega_2 = \int_{e_2}^\infty \frac{ds}{\sqrt{f(s)}}; \quad \wp\omega_2 = e_2, \quad \wp'\omega_2 = 0.$$

If further we write

$$f_1(s) = 4s^3 - g_2 s + g_3 = 4(s + e_1)(s + e_2)(s + e_3),$$

the quantity ω_2' is defined through the relations

$$(2) \qquad \frac{\omega_2'}{i} = \int_{-e_2}^\infty \frac{ds}{\sqrt{f_1(s)}}; \quad \wp\omega_2' = e_2, \quad \wp'\omega_2' = 0.$$

Note that ω_2' is a pure imaginary.

It may be shown that the three half-periods

$$\omega_1 = \tfrac{1}{2}(\omega_2 - \omega_2'), \quad \omega_2, \quad \omega_3 = \tfrac{1}{2}(\omega_2 + \omega_2'),$$

correspond to the quantities e_1, e_2, e_3 above, the relations being the same as are those for ω, $\omega'' = \omega + \omega'$, ω' in the usual notation where all three of the roots e_1, e_2, e_3 are real.

Write

$$\frac{d}{du}\zeta u = -\wp u,$$

* Fonctions Elliptiques, vol. 1, chap. 6.