A THEOREM ON SEMI-CONTINUOUS FUNCTIONS.

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RECENTLY G. C. Young* and A. Denjoy† have communicated theorems—those in Denjoy's memoir are of an especially comprehensive character—dealing, in particular, with point sets where the four derivatives of a given continuous function are identical. It is the purpose of this note to treat an analogous problem that arises when "derivative" is replaced by "saltus."‡ However, instead of confining ourselves tc "saltus," we prove a more general theorem that applies essentially to all semi-continuous functions.§ We preface the proof of this theorem with the following

LEMMA. *Let $f_1(x)$ and $f_2(x)$ be two real, single-valued functions, defined in the linear continuum, such that everywhere $f_1(x) \geqq f_2(x)$, and moreover, for every fixed real number k, the set S_k of points x where $f_1(x) \geqq k$ and $f_2(x) < k$ is countable. Then $f_1(x)$ and $f_2(x)$ are identical except at most in a countable set.*

Proof. Let $\{k_n\}$, $n = 1, 2, \cdots \infty$, be a set of k's everywhere dense in the linear continuum. The set|| $S = \mathfrak{S}(S_{k_n})$, which consists of all of the elements of every S_{k_n}, is also countable. We show that $f_1(x) = f_2(x)$ for every given x not in S. For let $\{k_{i_n}\}$ be a monotone decreasing sequence of k_n's having $f_2(x)$ as limit. Since x is given as not belonging to S, it must be that $f_1(x) < k_{i_n}$ for every n; for from $f_1(x) \geqq k_{i_n}$ and $f_2(x) < k_{i_n}$, we would conclude that x belonged to $S_{k_{i_n}}$ and

* *Acta Mathematica*, vol. 37 (1914), p. 141.

† *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 105.

‡ By the saltus (= oscillation) of a given function $f(x)$ at the point x we understand the greatest lower bound of the saltus of $f(x)$ in the interval (ξ, η), for all intervals (ξ, η) enclosing x as interior point. Cf. Hobson, The Theory of Functions of a Real Variable (1907), art. 180, and the author's paper, "Certain general properties of functions," *Annals of Mathematics*, vol. 18 (1917), p. 147. For the comprehensiveness of our result see the remark at the close, in conjunction with the theorem of this note and the corollaries.

§ "Essentially" in the sense that every semi-continuous function is exhibitable as the "associated" function $\phi(x)$ of a "monotone decreasing interval-function ϕ_{ab}" (see theorem below).

|| In regard to the notation, compare Hausdorff, Grundzüge der Mengenlehre (1914), p. 5.