CHANGING SURFACE TO VOLUME INTEGRALS.

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THE note of Dr. Poor on "Transformation theorems in the theory of the linear vector function" in this BULLETIN, January, 1916, page 174, raises the question: Why not make the work short by using other methods?

The equation* $\int d\boldsymbol{S}(\ \ ) = -\int d\tau\nabla(\ \ )$ is an obvious identity because

$$\int\int \boldsymbol{i}dydz(\ \ ) = -\int\int\int \boldsymbol{i}dydzdx\frac{\partial}{\partial x}(\ \ )$$

is merely a partial integration.

If Φ be a linear vector function,

$$\nabla(\Phi \cdot \boldsymbol{u}) = \nabla_{\Phi}(\Phi \cdot \boldsymbol{u}) + \nabla_{\mathbf{u}}(\Phi \cdot \boldsymbol{u}) = -\nabla_M(\Phi \cdot \boldsymbol{u}) + \nabla \boldsymbol{u} \cdot \Phi_C,$$

where the subscripts Φ and $\boldsymbol{u}$ mean that the differentiation affects only the function indicated and the subscript M means that the differentiation is with respect to the point M of which $\boldsymbol{u}$ is independent (other differentiations are with respect to P). Hence, integrating with no sign, with dot, and with cross,

$$\int d\boldsymbol{S}\Phi \cdot \boldsymbol{u} = \int d\tau\nabla_M(\Phi \cdot \boldsymbol{u}) - \int d\tau\nabla\boldsymbol{u} \cdot \Phi_C, \qquad \text{Theorem 3,}$$

$$\int d\boldsymbol{S} \cdot \Phi \cdot \boldsymbol{u} = \int d\tau\nabla_M \cdot (\Phi \cdot \boldsymbol{u}) - \int d\tau\nabla\boldsymbol{u} : \Phi, \qquad \text{Theorem 2,}$$

$$\int d\boldsymbol{S} \times \Phi \cdot \boldsymbol{u} = \int d\tau\nabla_M \times (\Phi \cdot \boldsymbol{u}) - \int d\tau(\nabla u \cdot \Phi_C)_\times, \qquad \text{Theorem 1.}$$

Next if $\Phi \cdot d\Psi = d\Psi \cdot \Phi$, then $d(\Phi \cdot \Psi) = d\Phi \cdot \Psi + d\Psi \cdot \Phi$ and $\nabla(\Phi \cdot \Psi) = \nabla\Phi \cdot \Psi + \nabla\Psi \cdot \Phi$. Hence on integrating, we have

$$\int d\boldsymbol{S}\Phi \cdot \Psi = -\int d\tau\nabla\Phi \cdot \Psi - \int d\tau\nabla\Psi \cdot \Phi, \qquad \text{not given,}$$

* Reference may be made to my review, "The unification of vectorial notations," this BULLETIN, vol. 16, May, 1910, p. 428, where I use $d\mathbf{S}$ as an exterior normal instead of an interior normal as here.