REMARKS ON FUNCTIONAL EQUATIONS.

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1. THE remarks of this article are in continuation of our paper in the BULLETIN, volume 19 (1912), pages 66–70. We have mentioned that a class of functional equations arises from the solution of the equation

$$\frac{\partial f(x, y)}{\partial x} + \mu(x, y)\frac{\partial f(x, y)}{\partial y} = 0, \tag{1}$$

where in particular we have assumed that $\mu(x, y) = \psi'(x)/\psi'(y)$. It is obvious that other classes of functional relations are obtained by considering the solution of the equation

$$\frac{dy}{dx} = \mu(x, y)$$

in connection with (1) when $\mu(x, y)$ is suitably specialized. The equation (1) may profitably be compared with another source of functional equations, that is, the equation*

$$\frac{\partial \varphi(x, y)}{\partial x} + \frac{\partial \varphi(x, y)}{\partial y} = \lambda(x, y). \tag{2}$$

The following functional relation is derivable from (1) and admits an explicit solution:†

$$\psi(x) - \psi(y) = \Omega^{-1}\{x\phi(y) - y\phi(x)\}. \tag{1'}$$

This equation can be solved in another manner by noticing that $\phi(x)$ can be formally eliminated. The result is

$$\Omega\{x - y\} = \Omega(x)\psi_1^{-1}(y) - \Omega(y)\psi_1^{-1}(x), \tag{1''}$$

where

$$\Omega(0) = 0, \quad \psi_1^{-1}(0) = 1,$$

$$\psi_1^{-1}(x) = \psi^{-1}\{x + \psi(1)\}.$$

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* Cf. V. Volterra, *Acc. Lincei*, vol. 19 (1910), pp. 169, 425; BULLETIN, vol. 19, p. 171; G. C. Evans, Proc. Cong. Math., Cambridge, 1913, vol. 1, p. 387; J. Hadamard, *L'Enseignement Mathématique*, vol. 14 (1912), p. 18, note; Paul Lévy, Paris thesis.

† Cf. BULLETIN, l. c., page 67.