A GENERALIZATION OF LINDELÖF'S THEOREMS ON THE CATENARY.

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THE object of the following note is to generalize two well known theorems on the catenary due to Lindelöf* by proving the following proposition:

Whenever the general integral of Euler's differential equation for the integral

$$J = \int f(x, y, y')dx \tag{1}$$

is of the form

$$y = \alpha\varphi\left(\frac{x-\beta}{\alpha}\right) \tag{2}$$

(*with* α, β *as constants of integration*), *the following two theorems hold:*

A) *If* A *and* A' *are a pair of conjugate points* (*in the wider sense*) *on an extremal for the integral* (1), *then the tangents to the extremal at* A *and* A' *meet at a point* T *of the x-axis, and vice versa.*†

B) *The value of the integral* J *taken along the arc* AA' *of the extremal is equal to the value of* J *taken along the broken line* ATA':‡

$$J(AA') = J(AT) + J(TA'). \tag{3}$$

The proof of the first theorem is almost immediate. For if

$$\mathfrak{E}_0: \qquad y = \alpha_0\,\varphi\left(\frac{x-\beta_0}{\alpha_0}\right) \tag{4}$$

be any particular extremal of the family (2) and $A(x_1, y_1)$ one of

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* Compare Lindelöf-Moigno, "Leçons sur le calcul des variations," pp. 209–213.

† Compare my "Vorlesungen über Variationsrechnung," p. 80.

‡ L. Bianchi has recently generalized Lindelöf's second theorem from the integral $\int y\sqrt{1+y'^2}dx$ to the more general integral $\int y^p\sqrt{1+y'^2}dx$, *Rendiconti della R. Accademia dei Lincei, Classe di scienze fisiche, matematiche e naturali*, series 5, vol. 19 (1910), p. 705. The extremals for this integral are of the form (2), so that Bianchi's result is contained as a special case in our theorem B).