NOTE ON A NEW NUMBER THEORY FUNCTION.

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THE present note deals with the properties of a number theory function defined by means of Euler's ϕ-function in the following way:

$\lambda(p^a) = \phi(p^a)$ when p is an odd prime;

$\lambda(2^a) = \phi(2^a)$ if $a = 0$, 1, or 2; $\lambda(2^a) = \frac{1}{2}\phi(2^a)$ if $a > 2$;*

$\lambda(2^a p_1^{a_1} \cdots p_i^{a_i}) =$ the lowest common multiple of $\lambda(2^a)$, $\lambda(p_1^{a_1})$, $\cdots$, $\lambda(p_i^{a_i})$, p_1, $\cdots$, p_i being different odd primes.

Throughout, in a congruence such as

$$x^a \equiv 1 \pmod{n}$$

it will be assumed that x is prime to n. Then we have the theorem

$$x^{\lambda(p^a)} \equiv 1 \pmod{p^a} \tag{1}$$

for every prime p and integer a. For, by Fermat's theorem, (1) is true when p is an odd prime and also when $p = 2$ and $a = 1$ or 2, in view of the definition of λ. Then we have to examine only the case where $p = 2$ and $a > 2$.

Now by Fermat's theorem we have

$$x^{\phi(2^a)} \equiv 1 \pmod{2^a}, \quad (a > 2).$$

But it is known that the foregoing congruence has no primitive root; that is, for any odd x the congruence is true when $\phi(2^a)$ is replaced by some factor of $\phi(2^a)$ less than the number itself. But $\frac{1}{2}\phi(2^a) = \lambda(2^a)$ is the largest factor of $\phi(2^a)$ less than itself and contains all other such factors. Then

$$x^{\frac{1}{2}\phi(2^a)} \equiv 1 \pmod{2^a}, \quad (a > 2).$$

Hence the theorem of congruence (1) is proved.

This result may be employed to obtain a simple demonstration of the following analog of Fermat's general theorem:

* It is in respect to this part of the definition alone that $\lambda(n)$ differs from $\psi(n)$ defined by Bachmann, Niedere Zahlentheorie, I, p. 157.