# A FUNDAMENTAL INVARIANT OF THE DISCONTINUOUS $\zeta$-GROUPS DEFINED BY THE NORMAL CURVES OF ORDER $n$ IN A SPACE OF $n$ DIMENSIONS.

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A NORMAL curve $C_n$ of order $n$ in a space $S_n$ of $n$ dimensions is transformed into itself by $\infty^3$ collineations

$$z_i' = \sum_{k=0}^{n} a_{ik} z_k \qquad (i = 0, 1, \cdots, n)$$

in $S_n$. Each of these collineations subjects the parameter $\zeta_1 : \zeta_2$ of $C_n$ to a linear substitution $\zeta_1' = \alpha\zeta_1 + \beta\zeta_2$, $\zeta_2' = \gamma\zeta_1 + \delta\zeta_2$. The group of collineations obtained by restricting the coefficients $a_{ik}$ to rational integral values with determinant $|a_{ik}| = 1$ is properly discontinuous, and the corresponding group $\Gamma_n$ of linear fractional substitutions

$$(1) \qquad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma = 1),$$

on the parameter $\zeta = \zeta_1 : \zeta_2$ is hence likewise properly discontinuous. This latter group $\Gamma_n$ we call the discontinuous $\zeta$-group defined by $C_n$.

The arithmetic definition of discontinuous groups of substitutions (1) forms one of the fundamental problems in the theory of automorphic functions. The above is an outline of one of the few effective methods that have been suggested for this purpose;* but on account of its complexity little has been done with it for values of $n > 2$.† General results applying to any value of $n$ are almost totally lacking. The case $n = 2$, however, has been exhaustively treated on the arithmetic side by Fricke,‡ and has yielded results of great importance. Funda-

* Fricke, Chicago Congress Papers, p. 85, and J. W. Young, "On a class of discontinuous $\zeta$-groups, etc.," *Rendiconti del Circolo matematico di Palermo*, vol. 23 (1907), p. 97.

† The case $n = 4$ has been partially treated by the author, loc. cit.

‡ Fricke, "Ueber indefinite Formen mit drei und vier Variabeln," *Göttinger Nachrichten*, Dec. 13, 1883. Cf. also Fricke-Klein, "Theorie der automorphen Functionen" (Leipzig, 1897), vol. 1, p. 502 ff.