# THE DETERMINATION OF THE CONSTANTS IN THE PROBLEM OF THE BRACHISTOCHRONE.*

BY PROFESSOR OSKAR BOLZA.

THE general solution of Euler's differential equation for the problem of the brachistochrone in a vertical plane is the doubly infinite system of cycloids †

$$\begin{aligned} x - x_0 + h &= \pm r(\omega - \sin \omega), \\ y - y_0 + k &= r(1 - \cos \omega), \end{aligned} \tag{1}$$

referred to a rectangular system of coördinates whose $(x, y)$-plane is the given vertical plane and whose positive $y$-axis is directed vertically downward; $x_0$, $y_0$ are the coördinates of the starting point $A$, $k$ is a given constant, viz.,

$$k = v_0^2/2g,$$

where $v_0$ is the initial velocity and $g$ the constant of gravity; finally $h$ and $r$ are the two constants of integration, and $r$ is essentially positive.

We suppose that the endpoints $A$ and $B$ are fixed and propose *to determine the constants of integration so that the cycloid*

*Very little attention seems to have been paid to the question of the determination of the constants in the problem of the brachistochrone. I have been able to find only the following few references on the subject in the literature of the calculus of variations:

Johann Bernoulli who first proposed the problem of the brachistochrone in 1696, and Jacob Bernoulli give a geometrical construction for the special case where the initial velocity is zero, in which case the starting point is a cusp of the cycloid (compare Ostwald's Klassiker der exacten Wissenschaften, no. 46, pp. 12 and 18).

Dienger, Grundriss der Variationsrechnung (1867), p. 38, reduces for the general case the determination of the constants to the two transcendental equations (7) and (8) of the text without entering into a further discussion of these equations.

Weierstrass in his lectures (1882) states without proof that it is always possible to construct a cycloid upon a given base passing through two given points $A$ and $B$, and only one cycloid which contains no cusp between $A$ and $B$.

† Compare for instance Lindelöf-Moigno, Calcul des variations, p. 228, and Pascal, Die Variationsrechnung, § 31, where numerous historical references are given.