THE CHARACTERIZATION OF COLLINEATIONS.

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A COLLINEATION is ordinarily defined as a point transformation which converts collinear points into collinear points, *i. e.*, one for which the family of ∞^2 straight lines, or the equivalent differential equation $y'' = 0$, is invariant. The question suggests itself whether this definition does not contain redundancies—whether it is not sufficient to require that only some straight lines shall remain straight. If we understand by a *simple* system of lines any system possessing the property that through each point of the region of the plane considered there passes one and only one line, then the result of this note may be stated:

If four simple systems of straight lines remain straight after a point transformation, then the same is necessarily true of all straight lines, and the transformation is, therefore, a collineation.

To prove this consider the general point transformation T

$$X = \phi(x, y), \quad Y = \psi(x, y), \tag{1}$$

where ϕ and ψ are one-valued continuous functions possessing first and second derivatives in the region considered. It is assumed, of course, that the Jacobian, $J = \phi_x\psi_y - \psi_x\phi_y$, does not vanish identically. The once extended transformation is

$$Y' = \frac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'}; \tag{2}$$

and the twice extended may be written *

$$Y'' = \frac{\alpha + \beta y' + \gamma y'^2 + \delta y'^3 + Jy''}{(\phi_x + \phi_y y'^3)}, \tag{3}$$

where

$$\begin{aligned} \alpha &= \phi_x\psi_{xx} - \psi_x\phi_{xx}, \\ \beta &= \phi_y\psi_{xx} - \psi_y\phi_{xx} + 2\phi_x\psi_{xy} - 2\psi_x\phi_{xy}, \\ \gamma &= \phi_x\psi_{yy} - \psi_x\phi_{yy} + 2\phi_y\psi_{xy} - 2\psi_y\phi_{xy}, \\ \delta &= \phi_y\psi_{yy} - \psi_y\phi_{yy}. \end{aligned} \tag{4}$$

* Lie-Scheffers, Continuirliche Gruppen, 1893, p. 33.