SINGULAR POINTS OF FUNCTIONS WHICH SATISFY PARTIAL DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE.

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IN the study of the nature of isolated singular points of harmonic functions of two variables * the following theorem may well be given a fundamental place:

I. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point* (x_0, y_0), *then u has the form*

$$(1) \qquad C \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + v(x, y),$$

where C is a constant and v is harmonic at (x_0, y_0).

This theorem follows at once from well known facts concerning functions of a complex variable.† It is, however, highly desirable to obtain some other proof for it in order to be able to follow out consistently the method introduced by Riemann of deducing the theory of functions of a complex variable from the theory of harmonic functions of two real variables. Such a proof I have recently found, and it turns out that it can be at once applied to large classes of partial differential equations which include Laplace's equation in two dimensions as a very special case.

The theorem thus generalized, together with some applications, forms the subject of the present paper.

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* I speak of a function of the n variables $x_1, \cdots x_n$ as harmonic at the point $(a_1, \cdots, a_n)$ if throughout the neighborhood of this point it has continuous partial derivatives of the first two orders and satisfies Laplace's equation $\Sigma\, \partial^2 u / \partial x_i^2 = 0$. I speak of it as harmonic throughout a region if it is harmonic at every point of the region. By an isolated singular point of a harmonic function I understand a point at which it fails to be harmonic, although it is harmonic at every other point in the neighborhood of this point.

† Cf. *Annals of Mathematics*, Second Series, Vol. I (1899), p. 38. The proof can be given most readily by noticing that the derivative of the function of the complex variable $x + yi$ of which u is the real part is single valued in the neighborhood of the point $x_0 + y_0 i$ and can therefore be developed about this point by Laurent's theorem. Integrating this series we have a development for the function of which u is the real part from which the theorem follows without difficulty.