

ON THE SUFFICIENT CONDITIONS IN THE
CALCULUS OF VARIATIONS.

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(Read before the American Mathematical Society, December 23, 1901.)

THE sufficient conditions in the calculus of variations have recently received a great deal of attention; * and it would seem fitting that attempts be made to simplify their discussion whenever possible, and to render the agreement more exact between the known necessary and the known sufficient conditions. Such is the purpose of this paper, which also seeks to present the sufficient conditions in compact form. The work will to a large extent follow lectures delivered at Göttingen by Professor Hilbert, 1899–1901.

1. HILBERT'S INVARIANT INTEGRAL.† WEIERSTRASS'S
SUFFICIENT CONDITION.

Let us consider a simple definite line integral

$$(1) \quad I = \int_{x_0}^{x_1} f(x, y, y') dx,$$

where

$$y' = \frac{dy(x)}{dx},$$

and where f is an analytic function of the three arguments x, y, y' , in a certain region R . Let us then restrict ourselves to the consideration of curves of integration contained in a realm B , consisting of curves of the type

$$(2) \quad y = \phi(x),$$

* Du Bois-Reymond, *Math. Annalen*, 15; Kneser, *Lehrbuch der Variationsrechnung*, and many memoirs in *Math. Annalen*; Osgood, *Annals of Math.*, 2d ser., vol. 2, no. 3, and *Transactions Amer. Math. Soc.*, vol. 2, pp. 166, 273; Whittemore, *Annals of Math.*, 2d ser., vol. 2, no. 3; Bolza, *Ithaca Colloquium* (summer meeting, Amer. Math. Soc., Aug., 1901), unpublished, and *Transactions Amer. Math. Soc.*, vol. 2, p. 422; Weierstrass, lectures at Berlin, 1879–1882, unpublished; Hilbert, lectures at Göttingen, 1899–1901, unpublished, etc.

† Compare Osgood, *Annals of Math.*, l. c., where Hilbert's proof is given.