CONCERNING THE ANGLES AND THE ANGULAR DETERMINATION OF PLANES IN 4-SPACE.

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I. *Introductory: Angles of Two Planes.*

IT has been established by Jordan * that, if two linear point spaces S_α and S_α', each of $n - \alpha$ dimensions, in a linear point space of n dimensions, have one and but one point in common, S_α and S_α' form α angles. In the special case where $n = 4$ and $\alpha = 2$, S and S' denote two planes of 4-space, and, as two planes in 4-space have one and in general but one common point, it follows that two planes of 4-space form two angles. That such is the case can be readily seen independently of the general theorem mentioned, in either of the following reciprocal ways: Two planes π and π' may be conceived as two flat pencils of lines having the common point P of the planes as common vertex. Of all the angles formed by lines of the one pencil with lines of the other there is a least angle θ, and this is not zero since by hypothesis the pencils have no line in common. Denote by π'' the plane determined by the lines l and l' where the latter belong respectively to π and π' and form the angle θ. As the planes π' and π'' have the line l' in common, they form an ordinary angle α. In order to bring π and π' into coincidence, it is sufficient to turn l' in π'' through the angle θ and then, as l and l' now coincide and π and π' have a common line, to rotate π or π' through the ordinary angle ω formed by π with π' in its new position. Reciprocally, π and π' may be regarded as pencils of lineoids † (ordinary 3-spaces). Either plane will accordingly be determined by any pair of its generating lineoids, *i.e.*, the lineoids enveloping or containing it. The planes π and π' have no common lineoid, for otherwise they would have, contrary to hypothesis, a common line. Accordingly there is a minimal angle φ, not zero, in the assemblage of angles formed by lineoids of the one plane with those of the other. Denote by π'' the plane determined by L and L', where L and L' are lineoids of π and π' respectively and

* Jordan: "Essai sur la géométrie à n dimensions," *Bull. de la Soc. Math. de France*, vol. 3, p. 129.

† Cf. Cole: "On rotations in space of four dimensions," *Amer. Jour. of Math.*, vol. 12.