ON THE GROUPS GENERATED BY TWO OPERATORS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, April 27, 1901.)

IT is well known that the only abelian groups which are generated by two operators are those which have just two independent generators.* If the order of such a group is p^m (p being any prime number), it must be of type $(m - a, a)$. In general, if the order of any one of these abelian groups is written in the form $2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ ($p_1, p_2, \cdots$ being odd prime numbers) it is clear that at least one of the subgroups of orders $2^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$ contains just two independent generators while the remaining ones (if any) are cyclic.

When the two generating operators are not commutative the matter becomes much more difficult. A very interesting and simple case presents itself when each of these operators (s_1, s_2) is of order two. From the equation $s_1 s_2 = s_3$ we obtain the following:

$$s_2 = s_1 s_3 \qquad s_2 s_1 = s_1 s_3 s_1 = (s_1 s_2)^{-1} = s_3^{-1} = s_2 s_3 s_2.$$

Since s_1 and s_2 transform s_3 into its inverse they generate the group of dihedral rotations.† By properly selecting s_1 and s_2 the order of s_3 can be made any positive integer whatever.‡ That is, *every group that is generated by two operators of order two is a dihedral rotation group and every dihedral rotation group is generated by two operators of order two.* When the order of s_3 is two this group is abelian and vice versa.

From the known theorems which we proceed to give, it follows almost directly that every symmetric group and every alternating group is generated by two operators. (1) If a primitive group whose degree exceeds 8 contains a substitution whose degree is less than 6, it is alternating or symmetric.§ (2) If a transitive group of degree n contains a substitution whose order is a prime number $p > n/a$ (a being the smallest factor of n) it is primitive. (3) If two

---

* Frobenius and Stickelberger, *Crelle*, vol. 86, 1879, p. 217. Only non-cyclic groups are considered in the present note.

† Cf. Klein, Ikosaeder, 1884, p. 9.

‡ American Journal of Mathematics, vol. 22, 1900, p. 185.

§ Netto-Cole, Theory of substitutions, 1892, p. 138; BULLETIN, vol. 4, 1898, p. 141.