SELECTED TOPICS IN THE GENERAL THEORY OF FUNCTIONS.

SIX LECTURES DELIVERED BEFORE THE CAMBRIDGE COLLOQUIUM, AUGUST 22–27, 1898.

BY PROFESSOR W. F. OSGOOD.*

Lecture I.

Picard's Theorem, and the Application of Riemann's Geometric Methods in the General Theory of Functions.

THE subject which I have chosen for the first lecture of the Colloquium is Picard's noted theorem which in its more restricted form† may be stated as follows: *Any function $G(z)$ which is single valued and analytic for all finite values of z takes on in general for at least one value of z any arbitrarily assigned value C. There may be one value, a, which the function does not take on. But if there is a second such value, b, the function reduces to a constant.*

To prove the theorem it is sufficient to establish the existence of a function $\omega(x)$ such that

(1) $\omega(x)$ is analytic for all but three values of x;

(2) $\omega(x)$ does not enter a certain region of the ω-plane, no matter what path x traces out in the x-plane.

For, let $x = G(z)$ and let the singular points of the function $\omega(x)$ be the points a, b, ∞. If z, starting with the value z_0, traces out a closed path in the z-plane, x, starting with the value x_0, will return to this value; but $\omega(x)$ may conceivably, when x describes this path, fail to return to its original value; *i. e.*, x may have described a path which *on the Riemann's surface* of the function $\omega(x)$ is not closed. To show that this is not the case, Picard reflects that, the path in the z-plane being drawn together continuously to a point, the corresponding path in the x-plane must behave likewise and hence in the course of its deformation cannot pass over any one of the points a, b, ∞. Hence $\omega(x)$, regarded as a function of z, is a single valued function, analytic for all finite values of z. Now by a well known theorem of Weier-

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† Picard, "Sur une propriété des fonctions entières," *Comptes Rendus*, vol. 88 (1879); also his Traité d'Analyse, vol. 2, p. 231.