EXAMPLE OF A SINGLE-VALUED FUNCTION WITH A NATURAL BOUNDARY, WHOSE INVERSE IS ALSO SINGLE-VALUED.

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(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THAT functions exist which are analytic within the unit circle, have the unit circle as a natural boundary, and take on no value more than once, can be readily shown.

Let T be a region of the $u + vi$-plane bounded by a single curve, the tangent of which turns continuously as the point of tangency traces out the curve ; then there exists, even in this case, which is more general than the cases considered by Schwarz and Neumann, a Green's function belonging to T,* so that the *interior* of T can be mapped conformally † on the *interior* of the unit circle. Furthermore, the *boundary* of T will, even in this case, go over into the *boundary* of the circle in such a manner as to render the transformation of the region consisting of T and its boundary on the region consisting of the circle and its boundary one-to-one and continuous.‡

Let the curve which bounds T be represented by the equations

$$u = \varphi(t), \quad v = \psi(t),$$

where φ, ψ denote continuous functions having the primitive period unity, so that, when t increases from t_0 to $t_0 + 1$, the point (u, v) describes the boundary once. The corresponding point (x, y) will then describe the unit circle once and the angle $\theta = \tan^{-1}\dfrac{y}{x}$ will be a single-valued, continuous function of t ; t, a single-valued, continuous function of θ.

* Poincaré's solution of the boundary value problem is sufficiently general to cover this case. Cf. Poincaré : "Sur les équations aux dérivées partielles de la physique mathématique," *Amer. Jour. of Math.*, vol. 12 (1890) ; Paraf's Thesis : "Sur le problème de Dirichlet et son extension au cas de l'équation linéaire générale du second ordre," Paris, 1892, and the *Toulouse Annales*, vol. 6. An account of these papers is given in Picard's Traité d'Analyse, vol. 2, ch. 4.

† Inaugural Dissertation, § 21; Göttingen, 1851.

‡ Cf. Painlevé : "Sur la théorie de la représentation conforme," *Comptes Rendus*, vol. 112 (1891), p. 653. This paper is based in part on his Thesis : "Sur les lignes singulières des fonctions analytiques," Paris, 1887, and the *Toulouse Annales*, vol. 2 (1888). Painlevé points out that even on the boundary angles are preserved.