ON AN EXTENSION OF SYLOW'S THEOREM.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at the Meeting of February 26, 1898.)

SINCE we shall employ Cauchy's theorem * in what follows it seems desirable to give a simple proof of it. It may be stated as follows: *A group G whose order g is divisible by a prime number p contains an operator of order p.*

We shall first suppose that G is Abelian. If it is generated by a single operator S of order np, we have $S^n \neq 1$ and $(S^n)^p = 1$. Hence S^n is the required operator. If G cannot be generated by a single operator we may represent a set of generating operators by $S_1, S_2, \cdots, S_r$. Since these generators are commutative the order of the group which any number of them generate cannot be divisible by any prime number that is not contained in the order of at least one of the generators. Hence the order of at least one of the given generators of G must be divisible by p, and some power of this generator must be the required operator of order p.

We may now suppose that G is a non-Abelian group of order np, and that our theorem is proved for all Abelian groups and for all non-Abelian groups whose orders are less than np. Let g_1 be the order of the largest subgroup of G that transforms a given operator into itself; $g \div g_1$ is the number of conjugates of this operator. Hence

$$g = \frac{g}{g_1} + \frac{g}{g_2} + \cdots + \frac{g}{g_k} \qquad \text{(A)}$$

k being the number of systems of conjugate operators and $g_1, g_2, \cdots, g_k$ being the orders of the largest subgroups that transform one operator of each system into itself. The operators for which $g_\alpha = g$ form an Abelian subgroup of G. If the order of this subgroup is not divisible by p some $g_\beta < g$ must be divisible by p, since the second member of (A) must be divisible by this number. The main features of this method of proof are due to Frobenius.

THEOREM I. *If a group G contains r $(r > 0)$ subgroups $G_1, G_2, \cdots, G_r$ of order $p_1^\alpha p_2^\beta p_3^\gamma \cdots$ ($p_1, p_2, p_3, \cdots$ being different*

---

* Cauchy: Exercises d'analyse, III (1844), p. 250. Cf. Jordan: Traité des substitutions (1870), p. 26.