QUATERNIONS AS NUMBERS OF FOUR-DIMENSIONAL SPACE.

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ONE of my students* has shown that quaternions may be extended to four-dimensional space, the generalized versor operation of a quaternion being that of turning a directed line so that its projections on two given "right-angled" planes describe given angles in definite senses. One of these planes of a quaternion is its ordinary plane in three dimensions; the other is that plane which is the locus of all lines through the origin perpendicular to the ordinary plane. Mr. Philip shows further that Hamilton's method of assigning directed lines as the "indices" of numbers is an extended Argand method in which the index of 1 is the fourth dimensional unit. Hamilton himself showed that the fourth proportional to three mutually perpendicular unit lines was "a species of fourth unit in geometry" to which the number 1 might be assigned, but he did not further carry out this geometrical idea.

I have found that when a line is turned as Philip describes the line itself describes a *plane* angle of the same magnitude. This fact leads to an interesting theory of parallel great circles of a four-dimensional sphere, *viz.*, great circles that are everywhere equally distant with respect to great arc measurements. Two and only two great circles may be drawn through a given point of the sphere parallel to a given great circle; of these one may be excluded by a definite convention, leaving one and only one proper parallel to a given great circle through a given point. Denoting the versor operation of a given quaternion by the directed arc of its turn then the turning value of this arc is unaltered by translation in its own direction or parallel to itself. The associative law of products becomes thus a matter of instantaneous proof, since we may move the great arc of any factor parallel to itself so that it begins where the preceding great arc ends.

The algebraic theory of quaternions, which is based upon the multiplication table of the units 1, i, j, k, and is inde-

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