ON CERTAIN METHODS OF STURM AND THEIR APPLICATION TO THE ROOTS OF BESSEL'S FUNCTIONS.

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LAST November Mr. M. B. Porter, a graduate student at Harvard, submitted to me a proof that when $n > -\frac{1}{2}$ the well known theorem that between two successive positive roots of $J_n(x)$ lies at least one root of $J_{n+1}(x)$ can be extended to give the theorem that between two successive positive roots of $J_n(x)$ lies *just* one root of $J_{n+1}(x)$.* This proof consisted in applying to Bessel's equation the following proposition due to Sturm:†

If in a certain interval of the x-axis $\varphi_1(x) < \varphi_2(x)$ then between two successive roots, lying in this interval, of a solution of the equation:

$$\frac{d^2y}{dx^2} = \varphi_1(x) \cdot y,$$

there cannot lie more than one root of a solution of the equation:

$$\frac{d^2y}{dx^2} = \varphi_2(x) \cdot y.$$

The application to Bessel's functions is immediate when we let $y = \sqrt{x}\, J_n(x)$ for then y satisfies the differential equation:

$$\frac{d^2y}{dx^2} = \left(\frac{4\,n^2-1}{4\,x^2} - 1\right)y.$$

Not being aware at the time that the theorem above quoted is explicitly given by Sturm, Mr. Porter gave a proof of it which depends upon some better known theorems of the same mathematician. This proof, which is different from the one given by Sturm, is reproduced below (p. 210).

There has just appeared in the *American Journal*, (vol. xix, p. 75) another proof of the theorem concerning the

* For the sake of simplicity of statement we confine our attention to positive roots, the negative roots being numerically equal to them.

† *Liouville's Journal*, vol. i., p. 136. We will in future quote this article by merely mentioning Sturm's name.