ON CAUCHY'S THEOREM CONCERNING COMPLEX INTEGRALS.

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THE following arrangement of the proof of Cauchy's theorem, that the integral of a holomorphic function in a simply connected region is a function of the limits of integration, but not of the path, is more elementary than the proofs ordinarily met with. Like the proofs given by Goursat * and Jordan,† it avoids the use of double integrals or the calculus of variations, and it has the advantage over these proofs not merely of brevity, but also of avoiding more or less complicated considerations of limits. The materials from which the following proof is built up are so familiar to all students of the theory of functions,‡ and the consequent probability that this method of putting them together is not unknown, is so great, that I should hardly have ventured to publish it, had it not seemed to possess peculiar pedagogical advantages. These depend, apart from its elementary character, on the fact that almost all the steps involved establish theorems, or illustrate methods, with which the student must become familiar sooner or later.

The theory of integrals of a function of a complex variable depends, as is well known, on the theory of integrals of the form

$$I = \int_{(a,b)}^{(x,y)} (Pdx + Qdy),$$

in which P and Q are real functions of the two real variables x, y. We will begin with the theorem

S being any region in the xy-plane in which P and Q are single-valued and have continuous first partial derivatives, a necessary and sufficient condition that the integral I should depend merely on the limits of integration is the existence in S of a single-valued function $\phi(x, y)$, for which $\dfrac{\partial \phi}{\partial x} = P$, $\dfrac{\partial \phi}{\partial y} = Q$.

For, if such a ϕ exists, $Pdx + Qdy$ is the complete differential of ϕ, and the integral I is the limit of the sum of the increments of ϕ, which we get in going along the path of integration from (a, b) to (x, y). But the sum of these increments is $\phi(x, y) - \phi(a, b)$, so that I is independent of the path of integration. Conversely, if I is independent of the path, it

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* See HARKNESS and MORLEY'S *Theory of Functions*, p. 164.

† *Cours d'Analyse* (2d ed.), vol. 1, p. 185.

‡ See, for instance, PICARD, *Traité d'Analyse*, vol. 1, pp. 81–83, and BOUSSINESQ, *Cours d'Analyse infinitésimal*, vol. 2, pp. 6–12.