ON THE CONVERGENCE OF THE SERIES USED IN THE SUBJECT OF PERTURBATIONS.

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THE perturbations of the planets and the coördinates of the moon have been developed by astronomers in infinite series of terms involving sines or cosines of linear functions of two or more arguments with positive or negative integral multipliers. These arguments vary proportionally with the time, and their periods, in accordance with notions derived from the theory of probabilities, are supposed to be incommensurable with each other. Recently M. Poincaré has much insisted that, under the latter condition, these series, in the rigorous mathematical sense, are divergent (*Les Méthodes Nouvelles de la Mécanique Céleste,* Vol. II., pp. 277–280). However, the reasons brought forward to sustain this opinion are scarcely convincing, and I think there has been some scepticism among astronomers in reference to the matter. Without attempting to find any flaw in M. Poincaré's logic, I simply wish to point out a class of cases where the convergency of the series can be shown in spite of the incommensurability of the component arguments.

In many problems of dynamics, where the integral of conservation of areas has place, we shall often have the longitude λ of the moving point given by a quadrature. We choose as our example the equation

$$\frac{1}{\mathrm{n}} \frac{d\lambda}{dt} = \sum_{i=0}^{i=\infty} \sum_{i'=-\infty}^{i'=+\infty} \alpha^{i+|i'|} \cos(il + i'l'), \tag{1}$$

in which $l = nt + c$ and $l' = n't + c'$, and α is a positive constant less than unity. Here λ corresponds to M. Poincaré's $\log x$ (p. 279 of the above-quoted volume). Under the condition named, the series of (1) is convergent. Now let both members of the equation be integrated; putting μ for $\frac{n'}{n}$, we have

$$\lambda = \epsilon + \mathrm{n}t + \frac{\mathrm{n}}{n} \Sigma \frac{1}{i + i'\mu} \alpha^{i+|i'|} \sin(il + i'l'), \tag{2}$$

ϵ being the added arbitrary constant, and the sign of summation Σ having the same extension as that of the double sign in (1), except that the combination $i = i' = 0$ is omitted. When