ON A CERTAIN CLASS OF CANONICAL FORMS.*

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AN interesting class of theorems occurs occasionally in the consideration of algebraical quantics, viz., when a quantic (or quantics) is not in general reducible to a form (or forms) which at first sight, when we count the number of constants involved, appears to be sufficiently general to admit of a finite number of reductions. Such cases bear an analogy to the porism in geometry, as the reductions are impossible, except when the quantic (or quantics) satisfies an invariant relation, and then the number of reductions is infinite. These cases are not common in binary quantics, and are not very remarkable when they do occur. An instance is: Let x_1, x_2, x_3 be three linear expressions in the variables. Then $x_1x_2x_3$ and $ax_1^3 + bx_2^3 + cx_3^3 + dx_1x_2x_3$ contain six constants, viz., three involved in x_1, x_2, x_3 and the three external constants $a / b / c / d$; but two general binary cubics which contain six constants cannot be simultaneously reduced to these forms, unless the combinantive invariant obtained by substituting differential symbols in one and operating on the other vanishes. This, however, is readily apparent from the fact that there is an identical linear relation connecting the three cubes x_1^3, x_2^3, x_3^3 and the product $x_1x_2x_3$, so that the second form is less general than it appears to be at first sight.

But in ternary and quaternary quantics there are several striking cases. The most remarkable, perhaps, is that discovered by Lüroth, viz, that a general plane quartic curve cannot be expressed linearly in terms of the fourth powers of five lines; say

$$a_1x_1^4 + a_2x_2^4 + a_3x_3^4 + a_4x_4^4 + a_5x_5^4. \qquad (1)$$

I insert here a proof of this result, as it involves a method which I propose to use farther on.

Consider the unique conic Σ which can be described to touch the five lines. Then substituting differential symbols, i.e., $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, for λ, μ, ν in the tangential equation $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$ of Σ, and operating with the resulting expression on (1), the remainder vanishes identically. For, operating with $(A, B, C, F, G, H)\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^2$ on $x_1^4 = (\alpha_1x + \beta_1y + \gamma_1z)^4$, we get a result proportional to $(A\alpha_1^2 + B\beta_1^2 + C\gamma_1^2 + 2F\beta_1\gamma_1 + 2G\gamma_1\alpha_1 + 2H\alpha_1\beta_1)x_1^2$; and this vanishes because x_1 touches Σ.

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