ON THE NUMBER OF INSCRIPTIBLE REGULAR POLYGONS.

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LET us denote for brevity the phrase, a regular polygon which is geometrically inscriptible, by P. The P's are few in number compared with the non-P's. The number of P's up to 100 is 24; up to 300 is 37; up to 1000 is 52; up to 1,000,-000, only 206. Indeed, a P must have for the number of its sides a *prime* number of the form $2^x + 1$, or the product of a power of 2 by any number of different primes of that form. This was proven by Gauss; and, more simply, by myself in another paper.* Further, x here must be a power of 2. For if x contains an odd factor m, such that $lm = x$, then will $2^x + 1$ be divisible by $2^l + 1$. This is seen by writing y for 2^l in $2^{lm} + 1$, which thus becomes $y^m + 1$, a number divisible by $y + 1$. But the inverse of this, that all numbers of the form $(2^{2^y} + 1)$ are prime, is not generally true, as Fermat affirmed. Euler pointed out that this rule [true for $y = 0$, 1, 2, 3, 4] fails for $y = 5$. Again, it is stated in Lucas' *Théorie des Nombres*, pages 51 and 448, that it fails for $y = 6$; but that we are still ignorant in the case of $y = 7$. Hence, the numbers $(2^{32} + 1)$ and $(2^{64} + 1)$ do not give P's; while $(2^{128} + 1)$ may or may not give a P. Thus the P's below $2^{32} + 1$ are given by 2^x times one, or 2^x times the product of two or more different ones, of the primes $2^1 + 1, 2^2 + 1, 2^4 + 1, 2^8 + 1, 2^{16} + 1$; that is, will fall under one of these 32 forms: 2^x, $3 \cdot 2^x$, $5 \cdot 2^x$, $3 \cdot 5 \cdot 2^x$, $17 \cdot 2^x$, $3 \cdot 17 \cdot 2^x$, $5 \cdot 17 \cdot 2^x$, $3 \cdot 5 \cdot 17 \cdot 2^x$, $257 \cdot 2^x$, $3 \cdot 257 \cdot 2^x$, . . . $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537 \cdot 2^x$. Further, all P's between $2^{32} + 1$ and $2^{128} + 1$ also fall under the same 32 forms; for the only *new* factors which could occur, $2^{32} + 1$ and $2^{64} + 1$, are ruled out, not being prime. We cannot proceed to P's of sides greater than $2^{128} + 1$ (which requires 39 figures to express it).

The object of this paper is to give a general expression for the number of inscriptible regular polygons between certain arbitrary limits.

Theorem I. *The number of P's below $2^x + 1$ sides, where x is less than* 32, *is* $\frac{(x-1)(x+2)}{2}$. This follows from the fact that the numbers below $2^{32} + 1$ giving P's have a definite

* See this volume of BULLETIN, p. 20.—ED.