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PERIODIC CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}^*$

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1. Introduction. A properly embedded surface Σ in $\mathbb{H}^2 \times \mathbb{R}$, invariant by a non-trivial discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$, will be called a periodic surface. We will discuss periodic minimal and constant mean curvature surfaces. At this time, there is little theory of these surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and other homogeneous 3-manifolds, with the exception of the space forms.

The theory of doubly periodic minimal surfaces (invariant by a $\mathbb{Z}^2$ group of isometries) in $\mathbb{R}^3$ is well developed. Such a surface in $\mathbb{R}^3$, not a plane, is given by a properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$, $\mathbb{T}$ some flat 2-torus. One main theorem is that a finite topology complete embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ has finite total curvature and one knows the geometry of the ends [11]. It is very interesting to understand this for such minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$, $\mathbb{M}^2$ a closed hyperbolic surface.

In this paper we will consider periodic surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The discrete groups G of isometries of $\mathbb{H}^2 \times \mathbb{R}$ we consider are generated by horizontal translations ϕ_l along geodesics γ of $\mathbb{H}^2$ and/or a vertical translation $T(h)$ by some $h > 0$. We denote by $\mathbb{M}$ the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G.

In the case G is the $\mathbb{Z}^2$ subgroup of the isometry group generated by ϕ_l and $T(h)$, $\mathbb{M}$ is diffeomorphic but not isometric to $\mathbb{T} \times \mathbb{R}$. Moreover $\mathbb{M}$ is foliated by the family of tori $\mathbb{T}(s) = (d(s) \times \mathbb{R})/G$ (here $d(s)$ is an equidistant to γ). All the $\mathbb{T}(s)$ are intrinsically flat and have constant mean curvature; $\mathbb{T}(0)$ is totally geodesic. In Section 3, we will prove an Alexandrov-type theorem for doubly periodic H-surfaces, i.e., an analysis of compact embedded constant mean curvature surfaces in such a $\mathbb{M}$ (Theorem 3.1).

The remainder of the paper is devoted to construct examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

The first example we want to illustrate is the singly periodic Scherk minimal surface. In $\mathbb{R}^3$, it can be understood as the desingularization of two orthogonal planes. H. Karcher [5] has generalized this to desingularize k planes of $\mathbb{R}^3$ meeting along a line at equal angles, these are called Saddle Towers. In $\mathbb{H}^2 \times \mathbb{R}$, two situations are similar to these examples: the intersection of a vertical plane with the horizontal slice $\mathbb{H}^2 \times \{0\}$ and the intersection of k vertical planes meeting along a vertical geodesic at equal angles. These surfaces, constructed in Section 4, are singly periodic and called,

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