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THE tt^* STRUCTURE OF THE QUANTUM COHOMOLOGY OF $\mathbb{C}P^1$ FROM THE VIEWPOINT OF DIFFERENTIAL GEOMETRY*

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1. Introduction. The quantum cohomology of $\mathbb{C}P^1$ provides a distinguished solution of the third Painlevé (PIII) equation. S. Cecotti and C. Vafa discovered this from a physical viewpoint (see [4], [5]). We shall derive this from a differential geometric viewpoint, using the theory of harmonic maps and in particular the generalized Weierstrass representation (DPW representation) for surfaces of constant mean curvature. The nontrivial aspects are the characterization of the solution, and its global behaviour. As yet, no treatment (including ours) could be described as completely satisfactory, but we hope that our viewpoint provides additional insight.

Mirror symmetry provides the context for this example: whereas the quantum cohomology of a Calabi-Yau manifold corresponds to a variation of Hodge structure, the quantum cohomology of a Fano manifold (such as $\mathbb{C}P^1$) should correspond to a variation of "semi-infinite Hodge structure" or "non-commutative Hodge structure" (see [1], [18], [24]). In both cases, the variation of Hodge structure can be described as a "tt^* structure". This originates from the physical notion of the ground state metric (Zamolodchikov metric) on a moduli space of supersymmetric field theories. It represents a fusion of topological (holomorphic) and anti-topological (anti-holomorphic) objects. In differential geometric terms, a tt^* structure can be described as a certain kind of pluriharmonic map.

In the language introduced by C. Hertling (see [18] and sections 10 and 11 of [19]) the essential point is that the quantum cohomology of $\mathbb{C}P^1$ gives rise to a TERP-structure which is pure and polarized. Such structures arise naturally in singularity theory, and an independent approach to the pure and polarized property for the mirror partner of $\mathbb{C}P^1$ (and other Fano manifolds) has been given by C. Sabbah in [29]. H. Iritani ([22]) described the result of Cecotti and Vafa for $\mathbb{C}P^1$ much more explicitly, from the mirror symmetry viewpoint. Our approach constitutes yet another formulation: it says that the extended harmonic map remains entirely within a single Iwasawa orbit of the loop group $\Lambda SU_{1,1}$.

We shall now sketch in more concrete terms the necessary background information. First of all, it is well known that the (small) quantum cohomology of $\mathbb{C}P^1$ is a commutative algebra $\mathbb{C}[b,q]/(b^2-q)$ which specializes to the ordinary cohomology algebra $\mathbb{C}[b]/(b^2)$ of $\mathbb{C}P^1$ when the value of the complex parameter q is set equal to zero. The quantum differential equation of $\mathbb{C}P^1$ is that given by the linear ordinary differential operator $(\lambda\partial)^2 - q$, where λ (often denoted by $\hbar$) is a complex parameter

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