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CALIBRATED ASSOCIATIVE AND CAYLEY EMBEDDINGS*

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Abstract. Using the Cartan-Kähler theory, and results on real algebraic structures, we prove two embedding theorems. First, the interior of a smooth, compact 3-manifold may be isometrically embedded into a G_2-manifold as an associative submanifold. Second, the interior of a smooth, compact 4-manifold K, whose double $doub(K)$ has a trivial bundle of self-dual 2-forms, may be isometrically embedded into a $Spin(7)$-manifold as a Cayley submanifold. Along the way, we also show that Bochner's Theorem on real analytic approximation of smooth differential forms, can be obtained using real algebraic tools developed by Akbulut and King.

Key words. Associative calibration, Cayley calibration.

AMS subject classifications. 53C25, 58A15

1. Introduction.

Let (M^7, g) be a Riemannian 7-manifold whose holonomy group $\mathrm{Hol}(g)$ is a subgroup of the exceptional group G_2. Then M is naturally equipped with a covariantly constant 3-form φ and 4-form $*\varphi$. We call (M, φ, g) a *G_2-manifold*. It is well known that φ and $*\varphi$ are *calibrations* on M, in the sense of Harvey and Lawson [12]. The corresponding calibrated submanifolds in M are called *associative 3-folds* and *coassociative 4-folds*, respectively.

Similarly, if (M^8, g) has $\mathrm{Hol}(g) \subseteq Spin(7)$, then M admits a covariantly constant, self-dual 4-form Ψ, and we call (M, Ψ, g) a *$Spin(7)$-manifold*. The 4-form Ψ is the *Cayley calibration*, and the calibrated submanifolds are *Cayley 4-folds*.

Constructing examples of manifolds with G_2 and $Spin(7)$ holonomy and their calibrated submanifolds is of interest because of their importance in string theory. Also, they provide new examples of volume minimizing submanifolds in a given homology class [12]. In [8], R. Bryant applied the Cartan-Kähler theory to show that: (1) every closed, real analytic, oriented Riemannian 3-fold can be isometrically embedded in a Calabi-Yau 3-fold as a special Lagrangian submanifold; and (2) every closed, real analytic, oriented Riemannian 4-fold with a trivial bundle of self-dual 2-forms can be isometrically embedded in a G_2-manifold as an coassociative submanifold. Moreover, the submanifolds above may be embedded as the fixed locus of a real structure (in the special Lagrangian case), or an anti G_2-involution (in the coassociative case).

In this paper, we will first show that Bryant's constructions can be repeated for the associative and Cayley submanifolds.

THEOREM 1.1. *Assume (K^3, g) is a closed, oriented, real analytic Riemannian 3-manifold. Then there exists a G_2-manifold (N^7, φ) and an isometric embedding $i : K \hookrightarrow N$ such that the image $i(K)$ is an associative submanifold of N. Moreover, (N, φ) can be chosen so that $i(K)$ is the fixed point set of a nontrivial G_2-involution $r : N \to N$.*

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