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INSTANTONS AND BRANES IN MANIFOLDS WITH VECTOR CROSS PRODUCTS*

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Abstract. In this paper we study the geometry of manifolds with vector cross products and its complexification.

First we develop the theory of instantons and branes and study their deformations. For example they are (i) holomorphic curves and Lagrangian submanifolds in symplectic manifolds and (ii) associative submanifolds and coassociative submanifolds in G_2-manifolds.

Second we classify Kähler manifolds with the complex analog of the vector cross product, namely they are Calabi-Yau manifolds and hyperkähler manifolds. Furthermore we study instantons, Neumann branes and Dirichlet branes on these manifolds. For example they are special Lagrangian submanifolds with phase angle zero, complex hypersurfaces and special Lagrangian submanifolds with phase angle $\pi/2$ in Calabi-Yau manifolds.

Key words. Instanton, Brane, vector cross product, complex vector cross product, Calibrated submanifold

AMS subject classifications. 37J05, 53C15, 53C25, 53C26, 53C38

1. Introduction. The vector product, or the cross product, in $\mathbb{R}^3$ was generalized by Gray ([1],[6]) to the product of any number of tangent vectors, called the *vector cross product* (abbrev. VCP). The list of Riemannian manifolds with VCP structures on their tangent bundles includes symplectic (or Kähler) manifolds, G_2-manifolds and *Spin* (7)-manifolds. They are manifolds with special holonomy and they play important roles in string theory, M-theory and F-theory respectively. In this paper we develop the geometry of the VCP in general.

We also introduce the complex analog of the VCP in definition 26, called the *complex vector cross product* (abbrev. $\mathbb{C}$-VCP). We prove that there are only two classes of manifolds with $\mathbb{C}$-VCPs.

THEOREM 1. *If M is a closed Kähler manifold with a $\mathbb{C}$-VCP, then M must be either (i) a Calabi-Yau manifold, or (ii) a hyperkähler manifold.*

They are again manifolds with special holonomy and they play important roles in Mirror Symmetry. Notice that the list in the Berger classification of holonomy groups of oriented Riemannian manifolds coincides with the list of Riemannian manifolds admitting a VCP or a $\mathbb{C}$-VCP.

We study the geometry of *instantons,*which are submanifolds in M preserved by the VCP. Instantons are always absolute minimal submanifolds in M. When an instanton is not a closed submanifold in M, we require its boundary to lie inside a *brane* in order to have a Fredholm theory for the free boundary value problem. For example, when M is a symplectic manifold, then instantons and branes are holomorphic curves and Lagrangian submanifolds in M respectively. These geometric objects play important roles in understanding the symplectic geometry of M. When M is a G_2-manifold (resp. *Spin* (7)-manifold), instantons correspond to BPS states in M-theory (resp. F-theory) and branes are supersymmetric cycles.

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