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004

PROJECTIVE HULLS AND THE PROJECTIVE GELFAND TRANSFORM*

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Dedicated, with affection and deep esteem, to the memory of S.-S. Chern.

Abstract. We introduce the notion of a **projective hull** for subsets of complex projective varieties parallel to the idea of a polynomial hull in affine varieties. With this concept, a generalization of J. Wermer's classical theorem on the hull of a curve in $\mathbf{C}^n$ is established in the projective setting. The projective hull is shown to have interesting properties and is related to various extremal functions and capacities in pluripotential theory. A main analytic result asserts that for any point x in the projective hull $\widehat{K}$ of a compact set $K \subset \mathbf{P}^n$ there exists a positive current T of bidimension (1,1) with support in the closure of $\widehat{K}$ and a probability measure μ on K with $dd^cT = \mu - \delta_x$. This result generalizes to any Kähler manifold and has strong consequences for the structure of $\widehat{K}$.

We also introduce the notion of a **projective spectrum** for Banach graded algebras parallel to the Gelfand spectrum of a Banach algebra. This projective spectrum has universal properties exactly like those in the Gelfand case. Moreover, the projective hull is shown to play a role (for graded algebras) completely analogous to that played by the polynomial hull in the study of finitely generated Banach algebras.

This paper gives foundations for generalizing many of the results on boundaries of varieties in $\mathbf{C}^n$ to general algebraic manifolds.

Key words. Polynomial hull, Gelfand Transformation, analytic varieties and their boundaries, Jensen measures, extremal functions, quasi-plurisubharmonic functions, pluripolar sets

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1. Introduction.

A beautiful classical theorem of John Wermer [W$_1$] states that the polynomial hull $\widehat{\gamma}_{\text{poly}}$ of a compact real analytic curve $\gamma \subset \mathbf{C}^n$, has the property that $\widehat{\gamma}_{\text{poly}} - \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbf{C}^n - \gamma$. (Recall that the polynomial hull of $K \subset\subset \mathbf{C}^n$ is the set of points $x \in \mathbf{C}^n$ such that $|p(x)| \leq \sup_K |p|$ for all polynomials p.)

This paper was largely motivated by the question:

Does there exist an analogous result for curves in complex projective space $\mathbf{P}^n$*?*

To this end we introduce the notion of the *projective hull* of a compact set $K \subset \mathbf{P}^n$. It is defined to be the set $\widehat{K}$ of points $x \in \mathbf{P}^n$ for which there exists a constant $C = C_x$ such that

$$\|\mathcal{P}(x)\| \;\leq\; C^d \sup_K \|\mathcal{P}\| \tag{1.1}$$

for all holomorphic sections $\mathcal{P}$ of $\mathcal{O}_{\mathbf{P}^n}(d)$ and all $d > 0$. Strong motivation for this definition comes from the fact (Prop. 2.3) that if γ is the boundary of a one-dimensional

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