ASIAN J. MATH.
Vol. 18, No. 3, pp. 439–464, July 2014

ON CERTAIN GENERALIZED HARDY'S INEQUALITIES AND APPLICATIONS*

DEMETRIOS A. PLIAKIS†

Key words. Hardy's inequality, normal crossings, resolvent expansion, singular asymptotics, local growth estimates.

AMS subject classifications. 35B45, 35J10.

Introduction. The classical inequality of Hardy for smooth functions $f \in C_0^\infty(f \in \mathbf{R} \setminus \{0\})$:

$$\int_{\mathbf{R}} x^s f^2 \leq \frac{4}{(s+1)^2} \int_{\mathbf{R}} x^{s+2} (f')^2$$

for $s \neq -1$ can be generalized in various ways and provides a weighted version of Poincaré's inequality. The standard generalizations replace the weight x^s with the radial variable or the boundary defining function of a smooth domain, are reduced to the one-dimensional case and are proved directly by partial integration, as far as the weight stays smooth. Here we we replace the weight by a homogeneous polynomial that is singular also away from the origin, so its zero set is a singular algebraic cone. In this case no direct method of the preceding form is available: the rectilinearization of such a set being non-trivial along its singularities. Specifically, singular algebraic varities are rectilinearized under the process of "resolution of singularities" then, their singularities unfold and appear as "normal crossings". We follow this procedure to the extent of " reduction of multiplicity " of an algebraic set and prove following generalization.

Let $P(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree d in n-real variables belonging to the class $\mathcal{P}^{gH}$ that we define in the next paragraph. Let $V(P) = \{x \in \mathbf{R}^n / P(x) = 0\}$ be the algebraic set that it defines. We introduce the Hardy factors:

$$\mathcal{H}^1(P) = P^{-\frac{2}{d}}, \quad \mathcal{H}^2(P) = \left| \frac{\nabla P}{P} \right|^2.$$

We prove the following generalized Hardy inequalities GHI_i:

$$\int_{\mathbf{R}^n} \mathcal{H}^i(P) f^2 \leq C_i(P) \int_{\mathbf{R}^n} |\nabla f|^2$$

for functions $f \in C_0^\infty(\mathbf{R}^n \setminus V(P))$. This inequality while it is elementary to prove when the algebraic variety $V(P)$ is smooth away from the origin, it is rather cumbersome when the variety is singular. The above inequality may be viewed as direct generalization of Hardy's Here, we will consider the stratification of the algebraic variety $V(P)$ by multiplicity and the inequality will be examined through the resolution of singularities process. This provides a finite covering, in every chart of which the algebraic set is reduced to normal crossings. The inequality is readily reduced to a corresponding one for inhomogeneous polynomials.

*Received November 29, 2010; accepted for publication March 6, 2013.
†TEI CRETE, Dept Electronics, Chania, Crete, Greece (dpliakis@gmail.gr).