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EIGENVALUES OF HECKE OPERATORS ON HILBERT MODULAR GROUPS*

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Abstract. Let F be a totally real field, let I be a nonzero ideal of the ring of integers $\mathcal{O}_F$ of F, let $\Gamma_0(I)$ be the congruence subgroup of Hecke type of $G = \prod_{j=1}^d \mathrm{SL}_2(\mathbb{R})$ embedded diagonally in G, and let χ be a character of $\Gamma_0(I)$ of the form $\chi\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = \chi(d)$, where $d \mapsto \chi(d)$ is a character of $\mathcal{O}_F$ modulo I.

For a finite subset P of prime ideals $\mathfrak{p}$ not dividing I, we consider the ring $\mathcal{H}^I$, generated by the Hecke operators $T(\mathfrak{p}^2)$, $\mathfrak{p} \in P$ (see §3.2) acting on (Γ, χ)-automorphic forms on G.

Given the cuspidal space $L^{2,\mathrm{cusp}}_\xi\big(\Gamma_0(I)\backslash G, \chi\big)$, we let V_ϖ run through an orthogonal system of irreducible G-invariant subspaces so that each V_ϖ is invariant under $\mathcal{H}^I$. For each $1 \le j \le d$, let $\lambda_\varpi = (\lambda_{\varpi,j})$ be the vector formed by the eigenvalues of the Casimir operators of the d factors of G on V_ϖ, and for each $\mathfrak{p} \in P$, we take $\lambda_{\varpi,\mathfrak{p}} \in \mathcal{J}_\mathfrak{p} := [0, 1 + \mathrm{N}(\mathfrak{p})) \cup i\big(0, \sqrt{1 + \mathrm{N}(\mathfrak{p})^2}\big]\big)$ so that $\lambda^2_{\varpi,\mathfrak{p}} - \mathrm{N}(\mathfrak{p})$ is the eigenvalue on V_ϖ of the Hecke operator $T(\mathfrak{p}^2)$. If for some prime $\mathfrak{p}$ the Hecke operator $T(\mathfrak{p})$ can be defined then its eigenvalue on V_ϖ is real and equal to $\lambda_{\varpi,\mathfrak{p}}$ or $-\lambda_{\varpi,\mathfrak{p}}$.

For each family of expanding boxes $t \mapsto \Omega_t$, as in (3) in $\mathbb{R}^d$, and fixed interval $J_\mathfrak{p}$ in $\mathcal{J}_\mathfrak{p}$, for each $\mathfrak{p} \in P$, we consider the counting function

$$N(\Omega_t;\, (J_\mathfrak{p})_{\mathfrak{p}\in P}) := \sum_{\varpi,\, \lambda_\varpi \in \Omega_t \,:\, \lambda_{\varpi,\mathfrak{p}} \in J_\mathfrak{p} \ ,\forall \mathfrak{p} \in P} |c^r(\varpi)|^2 .$$

Here $c^r(\varpi)$ denotes the normalized Fourier coefficient of order r at ∞ for the elements of V_ϖ, with $r \in \mathcal{O}'_F \smallsetminus \mathfrak{p}\,\mathcal{O}'_F$ for every $\mathfrak{p} \in P$.

In the main result in this paper, Theorem 1.1, we give, under some mild conditions on the Ω_t, the asymptotic distribution of the function $N(\Omega_t;\, (J_\mathfrak{p})_{\mathfrak{p}\in P})$, as $t \to \infty$. We show that at the finite places outside I the eigenvalues of the Hecke operator $T(\mathfrak{p}^2)$ are equidistributed compatibly with the Sato-Tate measure, whereas at the archimedean places the eigenvalues λ_ϖ are equidistributed with respect to the Plancherel measure.

As a consequence, if we pick an infinite place l and we prescribe $\lambda_{\varpi,j} \in \Omega_j$ for all infinite places $j \neq l$ and $\lambda_{\varpi,\mathfrak{p}} \in J_\mathfrak{p}$ for all finite places $\mathfrak{p}$ in P for fixed sets Ω_j and fixed intervals $J_\mathfrak{p} \subset \mathcal{J}_\mathfrak{p}$ with positive measure and then allow $\lambda_{\varpi,l}$ to run over larger and larger regions, then there are infinitely many representations ϖ in such a set, and their positive density is as described in Theorem 1.1.

Key words. Automorphic representations, Hecke operators, Hilbert modular group, Plancherel measure, Sato-Tate measure.

AMS subject classifications. 11F41, 11F60, 11F72, 22E30.

1. Introduction and discussion of main results. We work with a totally real number field F of degree d, the Lie group $G = \mathrm{SL}_2(\mathbb{R})^d$ considered as the product of $\mathrm{SL}_2(F_j)$ for all archimedean completions $F_j \cong \mathbb{R}$ of F. The group $\mathrm{SL}_2(F)$ is diagonally embedded in G. We consider the congruence subgroup $\Gamma = \Gamma_0(I)$ with I a nonzero ideal in the ring of integers $\mathcal{O}_F = \mathcal{O}$ of F, a character χ of $(\mathcal{O}_F/I)^*$ inducing a character $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)$ of Γ, and a compatible central character determined by $\xi \in \{0,1\}^d$.

$L^2_\xi(\Gamma\backslash G, \chi)$ is the Hilbert space of classes of square integrable functions transforming on the left by Γ according to the character χ, and transforming by the center Z of G according to the central character determined by ξ. We work with a maximal

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