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003

A SKELETON KEY TO ABHYANKAR'S PROOF OF EMBEDDED RESOLUTION OF CHARACTERISTIC P SURFACES*

STEVEN DALE CUTKOSKY†

Dedicated to Professor Hironaka, on the occasion of his 80th birthday

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This paper analyzes and simplifies Abhyankar's proof of embedded resolution of surface singularities in positive characteristic. Abhyankar's proof is obtained by combining the results in the papers [2], [4], [5], [6] and the first two chapters of the book [7]. This proof is extremely influential, but because of its length and complexity, is not generally well known and understood. In this article, I have written a report on the proof, hoping to make the main ideas more generally known. I give complete proofs of the essential parts of the proof. Some lemmas, which are given complete self contained proofs in Abhyankar's work, are merely stated and cited in this paper. Some of these cited results can be proven directly without great difficulty. I have made substantial simplifications in the original proofs, but have made a point of not making simplications which eliminate an original and interesting idea which could possibly have application to resolution in higher dimension.

Resolution of singularities in characteristic zero and in all dimensions was first proven by Hironaka [26]. More recently, there have been significant simplifications of this proof, including in [9], [11], [12], [20], [21], [28], [29], [34], [44], [45]. The first proof of resolution of surface singularities in characteristic $p > 0$ was by Abhyankar [1]. There have been other proofs of resolution of surface singularities in characteristic $p > 0$ since this time, including the proof analyzed in this paper, and proofs by Hironaka [27], Lipman [37], Hauser [24] and Cossart, Jannsen and Saito [16]. The first proof of resolution of singularities of 3-folds in positive characteristic $p > 5$ was given by Abhyankar [7], using the embedded resolution theorems for surfaces analyzed in this paper. A greatly simplified proof appears in [18], using Hironaka's algorithm [27] for embedded resolution of surface singularities. Recently, Cossart and Piltant have proven resolution of singularities of 3-folds in all characteristics [14], [15]. Some of the recent papers attacking resolution in higher dimensions and positive characteristic are [13], [19], [30], [31], [32], [33], [35], [39], [42], and [43].

Abhyankar's proof of embedded resolution of surface singularities is essentially a generalization of Zariski's characteristic zero proof [46] of embedded resolution of surface singularities. Zariski's final global proof, deducing the Theorem of Beppo Levi from local uniformization, extends without much difficulty to characteristic $p > 0$. The essential point where Zariski's proof does not extend to characteristic $p > 0$ is in local uniformization of a particular type of valuation ν which dominates a normal local domain of dimension two. This difficult case occurs when ν is rational nondiscrete. For the most part, for simplicity, we restrict to the analysis of this fundamental case.

The global argument used to deduce global resolution from local uniformization does not extend to dimension three, even in characteristic zero, as birational geometry

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†202 Mathematical Sciences Bldg, University of Missouri, Columbia, MO 65211, USA (cutkoskys@missouri.edu). Partially supported by NSF.