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007

GENERALIZED LAGRANGIAN MEAN CURVATURE FLOWS IN SYMPLECTIC MANIFOLDS*

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Abstract. An almost Kähler structure on a symplectic manifold (N, ω) consists of a Riemannian metric g and an almost complex structure J such that the symplectic form ω satisfies $\omega(\cdot, \cdot) = g(J(\cdot), \cdot)$. Any symplectic manifold admits an almost Kähler structure and we refer to (N, ω, g, J) as an almost Kähler manifold. In this article, we propose a natural evolution equation to investigate the deformation of Lagrangian submanifolds in almost Kähler manifolds. A metric and complex connection $\widehat{\nabla}$ on TN defines a generalized mean curvature vector field along any Lagrangian submanifold M of N. We study the evolution of M along this vector field, which turns out to be a Lagrangian deformation, as long as the connection $\widehat{\nabla}$ satisfies an Einstein condition. This can be viewed as a generalization of the classical Lagrangian mean curvature flow in Kähler-Einstein manifolds where the connection $\widehat{\nabla}$ is the Levi-Civita connection of g. Our result applies to the important case of Lagrangian submanifolds in a cotangent bundle equipped with the canonical almost Kähler structure and to other generalization of Lagrangian mean curvature flows, such as the flow considered by Behrndt [B] in Kähler manifolds that are almost Einstein.

Key words. Lagrangian mean curvature flow, almost Kähler structure, symplectic manifold, cotangent bundle.

AMS subject classifications. Primary 53C44.

1. Introduction.

Special Lagrangian submanifolds [HL] and Lagrangian mean curvature flows [TY] attract much attentions due to their relations to the SYZ conjecture [SYZ] on mirror symmetry between Calabi-Yau manifolds. A Calabi-Yau, or in general a Kähler-Einstein manifold, is a great place to study the mean curvature flow as this process provides a Lagrangian deformation [SM1]. This important property no longer holds if the ambient space is a general symplectic manifold. However, there are important conjectures (see for example [FSS]) concerning the Lagrangian isotopy problem in general symplectic manifolds such as cotangent bundles which do not carry Kähler-Einstein structures.

In this article, we aim at defining a generalized Lagrangian mean curvature flow in general almost Kähler manifolds N. We consider generalized mean curvature vector fields $\overrightarrow{\widehat{\mathrm{H}}}$ (see Definition 3 in §4) along Lagrangian or more generally almost Lagrangian submanifolds, i.e. submanifolds M for which $J(TM) \cap TM = \{0\}$. The definition of the generalized mean curvature vector $\overrightarrow{\widehat{\mathrm{H}}}$ relies on a choice of a complex and metric connection $\widehat{\nabla}$ on TN that could carry non-trivial torsion $\widehat{T}$. We then say that a smooth family of almost Lagrangian immersions

$$F : M \times [0, T) \to N$$

satisfies the generalized mean curvature flow, if

$$\frac{\partial F}{\partial t}(p, t) = \overrightarrow{\widehat{\mathrm{H}}}(p, t), \quad \text{and} \quad F(M, 0) = M_0 \tag{1}$$

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