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MULTIFRACTAL ANALYSIS FOR CONVOLUTIONS OF OVERLAPPING CANTOR MEASURES*

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Abstract. Unlike the case for self-similar measures satisfying the open set condition, it has been shown that the m-fold convolution of the uniform Cantor measure on the classical middle-third Cantor set has isolated points in its multifractal spectrum for any $m \geq 3$. We show that this phenomena of isolated points holds for quite general Cantor measures on Cantor sets that can be far from self-similar.

We also prove, in contrast, that if the convolution is understood on the group $[0, 1]$, rather than on $\mathbb{R}$, then the multifractal spectrum of the 3-fold convolution of the uniform Cantor measure is an interval.

Key words. Multifractal analysis, local dimension, Cantor measure, convolution.

AMS subject classifications. Primary 28A80, 28A78; Secondary 42A36.

1. Introduction. A useful tool in the study of singular measures is the concept of the local dimension of the measure μ defined at points x in the support of μ by

$$\dim_{loc} \mu(x) = \lim_{r \to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

For measures that are suitably uniform the local dimension can be the same value at every point in the support of the measure, but for more general measures it is of interest to determine for which α the sets $E_\alpha = \{x : \dim_{loc} \mu(x) = \alpha\}$ are non-empty, the so-called multifractal spectrum, and to quantify the size of these sets. This has been done for many examples of measures, including (quasi) self-similar measures and p-Cantor measures on central Cantor sets which satisfy a separation condition (the open set condition in the case of self-similar measures). For such measures it is known that the multifractal spectrum is an interval and there is a formula for calculating the dimensions of the sets E_α, known as the multifractal formalism (c.f., [2], [6], [9], [10]).

The uniform Cantor measure, μ, supported on the classical middle-third Cantor set, and its m-fold convolutions, denoted μ^m, are interesting examples of self-similar measures generated by the iterated function systems (IFS) $\{F_i(x) = x/3+2i/3\}$ acting on $[0, m]$, with probabilities $\{2^{-m}\binom{m}{i}\}$, for $i = 0, 1, \ldots, m$. The Cantor measure μ has the same local dimension at all points of its support. When $m = 2$, the open set condition is satisfied and the multifractal spectrum of μ^2 can be obtained through the multifractal formalism.

However, if $m \geq 3$ the open set condition does not hold and in [7] Hu and Lau discovered the striking fact that the multifractal spectrum of μ^m is not an interval. In fact, they showed that $\dim_{loc} \mu^m(0)$ is an isolated point in the multifractal spectrum and is the maximum local dimension.

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