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GOOD SHADOWS, DYNAMICS AND CONVEX HULLS OF COMPLETE SUBMANIFOLDS*

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Abstract. Any non-empty open convex subset of $\mathbb{R}^n$ is the convex hull of a complete submanifold M, of any codimension, but there are obstructions if the geometry of M is, *a priori*, suitably controlled at infinity. In this paper we develop tools to explore the geometry of $\partial[\text{Conv}(M)]$ when the Grassmanian-valued Gauss map of M is *uniformly continuous*, a condition that, in the C^2 case, is weaker than bounding the second fundamental form of M. Our proofs are based on the Ekeland variational principle, and on a conceptual refinement of the Omori-Yau asymptotic maximum principle that is of interest in its own right. If the Ricci (sectional) curvature of M is bounded below and f is a C^2 function on M that is bounded above, then not only there exists *some* maximizing sequence for f with good properties, as predicted by the Yau (Omori) principle but, in fact, *every* maximizing sequence for f can be *shadowed* by a maximizing sequence that has good properties. This abundance of *good shadows* allows for *information to be localized at infinity,* revealing in our geometric setting the relation between the asymptotic behavior of M and the supporting hyperplanes of $\partial[\text{Conv}(M)]$ in general position that pass through some fixed boundary point. We also introduce a new approach to asymptotic maximum principles, based on dynamics, to prove a special case of a conjecture meant to extend our refinement of the Yau maximum principle to manifolds that satisfy a property weaker than $\inf \text{Ric} > -\infty$. The authors expect that this new understanding of the Omori-Yau principle – in terms of good shadows and localization at infinity – will lead to applications in contexts other than convexity.

Key words. Omori-Yau maximum principles, localization at infinity, convex hulls, isometric immersions.

AMS subject classifications. 35B50, 53C42, 53C21, 37C10.

1. Introduction.

Any non-empty open convex subset $\mathcal{O}$ of $\mathbb{R}^n$ is the convex hull of a C^∞ complete submanifold M, of any codimension. To see this when $n \geq 3$, take a smooth embedded curve $\Gamma \subset \mathcal{O}$, of infinite length on both ends, whose convex hull is $\mathcal{O}$. Let M be the union over all $p \in \Gamma$ of smoothly varying k-dimensional spheres $S^{(k)}_{r(p)}$, $1 \leq k \leq n-2$, centered at p and contained in the normal space of Γ at p. Taking $r(p)$ to decay fast enough one can make sure that the resulting manifold M, which is automatically complete, is contained in $\mathcal{O}$. Since p is in the convex hull of $S^{(k)}_{r(p)}$ for any $p \in \Gamma$, it follows that the convex hull of M satisfies $\text{Conv}\,(M) = \mathcal{O}$.

Despite the examples of the previous paragraph, one expects that not every $\mathcal{O}$ can be realized as $\text{Conv}\,(M)$ if the complete submanifold M has a geometry that is, *a priori*, suitably controlled at infinity. More generally, we study $\partial[\text{Conv}h(M)]$, where $h : M \to \mathbb{R}^n$ is an immersion, $\dim M = m$, and the induced metric is complete. Along the way, we introduce new tools that may be useful in other problems as well.

A natural way to control the geometry of a submanifold is to bound its second fundamental form, but this requires the immersion to be at least of class C^2. Instead, we work here with a weaker condition that makes sense even in the C^1 case: the Grassmanian-valued Gauss map $\mathcal{G} : M \to G(n-m, n)$, given by $\mathcal{G}(p) = [h_* T_p M]^\perp$, is *uniformly continuous.* Indeed, if the immersion is C^2 then boundedness of the second fundamental form is equivalent to the Gauss map being Lipschitz, a condition that

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