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ON SOME PARTITIONS OF A FLAG MANIFOLD*

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Dedicated to Professor Dan Papuc on his 80th birthday

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Introduction. Let G be a connected reductive group over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Let $\mathbf{W}$ be the Weyl group of G. Let $\underline{\mathbf{W}}$ be the set of conjugacy classes in $\mathbf{W}$. The main purpose of this paper is to give a (partly conjectural) definition of a surjective map from $\underline{\mathbf{W}}$ to the set of unipotent classes in G (see 1.2(b)). When $p = 0$, a map in the opposite direction was defined in [KL, 9.1] and we expect that it is a one sided inverse of the map in the present paper. The (conjectural) definition of our map is based on the study of certain subvarieties $\mathcal{B}_g^w$ (see below) of the flag manifold $\mathcal{B}$ of G indexed by a unipotent element $g \in G$ and an element $w \in \mathbf{W}$.

Note that $\mathbf{W}$ naturally indexes ($w \mapsto \mathcal{O}_w$) the orbits of G acting on $\mathcal{B} \times \mathcal{B}$ by simultaneous conjugation on the two factors. For $g \in G$ we set $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$. The varieties $\mathcal{B}_g$ play an important role in representation theory and their geometry has been studied extensively. More generally for $g \in G$ and $w \in \mathbf{W}$ we set

$$\mathcal{B}_g^w = \{B \in \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}.$$

Note that $\mathcal{B}_g^1 = \mathcal{B}_g$ and that for fixed g, $(\mathcal{B}_g^w)_{w \in \mathbf{W}}$ form a partition of the flag manifold $\mathcal{B}$.

For fixed w, the varieties $\mathcal{B}_g^w$ ($g \in G$) appear as fibres of a map to G which was introduced in [L3] as part of the definition of character sheaves. Earlier, the varieties $\mathcal{B}_g^w$ for g regular semisimple appeared in [L1] (a precursor of [L3]) where it was shown that from their topology (for $\mathbf{k} = \mathbf{C}$) one can extract nontrivial information about the character table of the corresponding group over a finite field.

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1. The sets $\mathbf{S}_g$.

1.1. We fix a prime number l invertible in $\mathbf{k}$. Let $g \in G$ and $w \in \mathbf{W}$. For $i, j \in \mathbf{Z}$ let $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j$ be the subquotient of pure weight j of the l-adic cohomology space $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)$. The centralizer $Z(g)$ of g in G acts on $\mathcal{B}_g^w$ by conjugation and this induces an action of the group of components $\bar{Z}(g)$ on $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)$ and on each $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j$. For $z \in \bar{Z}(g)$ we set

$$\Xi_{g,z}^w = \sum_{i,j \in \mathbf{Z}} (-1)^i \mathrm{tr}(z, H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j) v^j \in \mathbf{Z}[v]$$

where v is an indeterminate; the fact that this belongs to $\mathbf{Z}[v]$ and is independent of the choice of l is proved by an argument similar to that in the proof of [DL, 3.3].

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