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WACH MODULES AND IWASAWA THEORY FOR MODULAR FORMS*

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Dedicated to John Coates, on the occasion of his 66th birthday

Abstract. We define a family of Coleman maps for positive crystalline p-adic representations of the absolute Galois group of $\mathbb{Q}_p$ using the theory of Wach modules. Let $f = \sum a_n q^n$ be a normalized new eigenform and p an odd prime at which f is either good ordinary or supersingular. By applying our theory to the p-adic representation associated to f, we define Coleman maps $\underline{\mathrm{Col}}_i$ for $i = 1, 2$ with values in $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Z}_p} \Lambda$, where Λ is the Iwasawa algebra of $\mathbb{Z}_p^\times$. Applying these maps to the Kato zeta elements gives a decomposition of the (generally unbounded) p-adic L-functions of f into linear combinations of two power series of bounded coefficients, generalizing works of Pollack (in the case $a_p = 0$) and Sprung (when f corresponds to a supersingular elliptic curve). Using ideas of Kobayashi for elliptic curves which are supersingular at p, we associate to each of these power series a Λ-cotorsion Selmer group. This allows us to formulate a "main conjecture". Under some technical conditions, we prove one inclusion of the "main conjecture" and show that the reverse inclusion is equivalent to Kato's main conjecture.

Key words. L-function, supersingular modular form, p-adic Hodge theory, Iwasawa theory.

AMS subject classifications. 11R23,11F80,11S31,12H25

1. Introduction.

1.1. Background. Let E be an elliptic curve defined over $\mathbb{Q}$ which has good ordinary reduction at the prime p. In [MSD74], Mazur and Swinnerton-Dyer constructed a p-adic L-function, $\tilde{L}_{p,E}$, which interpolates complex L-values of E. Let $\mathbb{Q}_\infty = \mathbb{Q}(\mu_{p^\infty})$. If G_∞ denotes the Galois group of $\mathbb{Q}_\infty$ over $\mathbb{Q}$, then $\tilde{L}_{p,E}$ is an element of $\Lambda_{\mathbb{Q}_p}(G_\infty) = \mathbb{Q} \otimes \mathbb{Z}_p[[G_\infty]]$. It is conjectured that $\tilde{L}_{p,E}$ is in fact an element of the Iwasawa algebra $\Lambda(G_\infty) = \mathbb{Z}_p[[G_\infty]]$.

Recall that the p-Selmer group of E over any finite extension F of $\mathbb{Q}$ is defined as

$$\mathrm{Sel}_p(E/F) = \ker\left(H^1(F, E_{p^\infty}) \longrightarrow \prod_v \frac{H^1(F_v, E_{p^\infty})}{E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right),$$

where the product is taken over all places of F. If we let $\mathrm{Sel}_p(E/\mathbb{Q}_\infty) = \varinjlim_n \mathrm{Sel}_p(E/\mathbb{Q}(\mu_{p^n}))$, then $\mathrm{Sel}_p(E/\mathbb{Q}_\infty)$ is equipped with an action of G_∞ which extends to an action of the Iwasawa algebra. It is not difficult to show that the Pontryagin dual $\mathrm{Sel}_p(E/\mathbb{Q}_\infty)^\vee$ is finitely generated over $\Lambda(G_\infty)$, and a theorem of Kato-Rohrlich (conjectured by Mazur) states that it is in fact $\Lambda(G_\infty)$-torsion. We can therefore associate to it a characteristic ideal for each Δ-isotypical component,

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