ASIAN J. MATH.
Vol. 14, No. 2, pp. 243–256, June 2010

005

QUASI-FUCHSIAN 3-MANIFOLDS AND METRICS ON TEICHMÜLLER SPACE*

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Abstract. An almost Fuchsian 3-manifold is a quasi-Fuchsian manifold which contains an incompressible closed minimal surface with principal curvatures in the range of $(-1, 1)$. By the work of Uhlenbeck, such a 3-manifold M admits a foliation of parallel surfaces, whose locus in Teichmüller space is represented as a path γ, we show that γ joins the conformal structures of the two components of the conformal boundary of M. Moreover, we obtain an upper bound for the Teichmüller distance between any two points on γ, in particular, the Teichmüller distance between the two components of the conformal boundary of M, in terms of the principal curvatures of the minimal surface in M. We also establish a new potential for the Weil-Petersson metric on Teichmüller space.

Key words. Quasi-Fuchsian manifolds, foliation, minimal surfaces, Teichmüller distance, Weil-Petersson metric.

AMS subject classifications. Primary 30F60, 32G15; Secondary 53C42, 57M50

1. Introduction of Main Results. One of the fundamental questions in hyperbolic geometry of two and three dimensions is the interaction between the internal geometry of a hyperbolic 3-manifold and the geometry of Teichmüller space of Riemann surfaces. It is natural to consider the situation for complete quasi-Fuchsian 3-manifolds. Let M be such a manifold, then M is topologically identified as $M = \Sigma \times \mathbb{R}$, where Σ is a closed surface of genus $g \geq 2$. We denote the Teichmüller space of genus g surfaces by $\mathcal{T}_g(\Sigma)$, the space of conformal structures on Σ modulo orientation preserving diffeomorphisms in the homotopy class of the identity map. An important theorem of Brock ([Bro03]), proving a conjecture of Thurston, states that the hyperbolic volume of the convex core of a quasi-Fuchsian 3-manifold is quasi-isometric to the Weil-Petersson distance between the two components of the conformal boundary of the 3-manifold in Teichmüller space.

The space of quasi-Fuchsian 3-manifolds $QF(\Sigma)$, called *the quasi-Fuchsian space*, is a complex manifold of complex dimension $6g-6$. Its geometrical structures are extremely complicated, and they have been subjects of intensive study in recent years. In this paper, we consider mostly a subspace formed by the so-called *almost Fuchsian 3-manifolds*: a quasi-Fuchsian 3-manifold M is almost Fuchsian if it contains a unique embedded minimal surface, representing the fundamental group, whose principal curvatures are in the range of $(-1, 1)$. This subspace is an open connected manifold of the same dimension ([Uhl83]). One can view the quasi-Fuchsian space as a "higher" Teichmüller space, a square with Teichmüller space sitting inside as a diagonal, and view the space of almost Fuchsian 3-manifolds as an open subspace of $QF(\Sigma)$ around this diagonal. See also ([KS07]) for generalization of almost Fuchsian manifolds to dS, AdS geometry and surfaces with cone points.

By a remarkable theorem of Uhlenbeck ([Uhl83]), any almost Fuchsian 3-manifold admits a foliation of parallel surfaces from the unique minimal surface. These parallel

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