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AUTODUAL CONNECTION IN THE FOURIER TRANSFORM OF A HIGGS BUNDLE*

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Abstract. The Fourier transform of a stable Higgs bundle of degree zero and rank ≥ 2 on a curve of genus ≥ 2 is a locally free sheaf on the cotangent bundle $T^*\mathrm{J}(X)$ of the Jacobian of the curve. The base $T^*\mathrm{J}(X)$ can be identified with the fine moduli space of Higgs bundles of rank 1 and degree 0 on X and carries a natural hyper-Kähler structure.

We use a combination of generalised $\mathcal{D}$-module theory and twistor methods to show that the Fourier transform admits a natural connection which is autodual with respect to the hyper-Kähler structure. The construction exploits Deligne's and Simpson's description of the twistor space of the base in terms moduli spaces of λ-connections.

Key words. Higgs bundle, Fourier transform, Nahm transform, autodual connection, hyperholomorphic, hyper-Kahler, twistor space, D-module.

AMS subject classifications. 14H60, 53C07; Secondary 14F10, 32C38, 53C26, 53C28

1. Introduction.

Let X be a smooth complete algebraic curve over an algebraically closed field k (i.e., a compact Riemann surface in case $k = \mathbf{C}$), and let $\mathsf{E} = (\mathcal{E}, \theta)$ be a stable Higgs bundle of degree zero on X. In [4] we introduced a Fourier-Mukai transform $\widehat{\mathsf{E}}$ of E, which is an algebraic vector bundle on the cotangent bundle $T^*\mathrm{J}(X)$ of the Jacobian of X, and proved that it extends to an algebraic vector bundle on projectivisation of the base.

Following a suggestion of N.J. Hitchin, one expects that in the case where $k = \mathbf{C}$ this Fourier transform should carry a natural connection, thus completing the construction of an essentially *Nahm-type* transformation for Higgs bundles on curves. Taking cue from Jardim [15], one expects that the connection should be autodual in a suitable sense, corresponding to the doubly-periodic instantons in the genus-1 case. This paper confirms these expectations and completes the construction of the Nahm-type transformation announced in [4] by exhibiting a construction of a connection in the Fourier transform with the expected properties. The results appearing here have been originally proved in the author's thesis [3].

The base manifold $T^*\mathrm{J}(X)$ of the Fourier transform $\widehat{\mathsf{E}}$ can be identified with the fine moduli space $\mathbf{M}_{Dol}(X, 1)$ of Higgs bundles of rank 1 and degree 0 on X. By Hitchin [12] it carries a natural hyper-Kähler structure: it is Kähler with respect to three complex structures I, J and K satisfying $IJK = -1$, with I being the complex structure of the Higgs bundle moduli space and J, K being associated to the same underlying set as the moduli space of flat line bundles on X.

We show that $\widehat{\mathsf{E}}$ admits a natural connection ∇, which is autodual with respect to this natural hyper-Kähler structure: its curvature is invariant under the action of the group $\mathbf{SU}(2)$ of unit quaternions, specified by the actions of the three complex structures I , J and K on the tangent bundle. Equivalently, the connection is hyperholomorphic, or has curvature of type $(1, 1)$ with respect to all the complex structures in the hyper-Kähler family of the base (Kaledin-Verbitsky [16]).

Using a result of Kaledin and Verbitsky [16] (generalising one from Capria-Salamon [6]), we reduce the construction of the autodual connection to the construc-

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